# On some non self-adjoint problems in superconductivity theory.

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# Abstract

This work was initially motivated by a paper of Yaniv Almog at Siam J. Math. Appl. [Alm2]. The main goal is to show how some non self-adjoint operators appear in a specific problem appearing in superconductivity, to analyze their spectrum (in particular the non emptyness), their pseudo-spectrum and the decay of the associated semi-group. These results are obtained together with Y. Almog and X. Pan [AlmHelPan1, AlmHelPan2, AlmHelPan3]. These pseudo-spectral methods appear also in the analysis of the Fokker-Planck equation.

# Outline

1. Abstract Analysis : from resolvent estimates to decay estimates for the semi-group.
2. A toy model : Airy's operator.
3. The problem in superconductivity.
4. The Fokker-Planck operator.

# ABSTRACT ANALYSIS

# From resolvent estimates to decay estimates for the semi-group

**Theorem GP1 : Gearhart-Prüss Theorem**

Let $A$ be a closed operator with dense domain $D(A)$ generating a strongly continuous semi-group $T(t) = e^{tA}$ and $\omega \in \mathbb{R}$. Assume that $\|(z - A)^{-1}\|$ is uniformly bounded in the half-plane $\mathrm{Re}\, z \geq \omega$. Then there exists a constant $M > 0$ such that $P(M, \omega)$ holds:

$$\|T(t)\| \leq Me^{\omega t}\,.$$

# Reformulation in terms of $\epsilon$-spectra

For any $\epsilon > 0$, we define the $\epsilon$-spectra by

$$\sigma_\epsilon(\mathcal{A}) = \{z \in \mathbb{C}\,,\, ||(z - \mathcal{A})^{-1}|| > \frac{1}{\epsilon}\}\,.$$

For a given accretive closed operator $\mathcal{A} = -A$, we introduce

$$\widehat{\alpha}_\epsilon(\mathcal{A}) = \inf_{z \in \sigma_\epsilon(\mathcal{A})} \operatorname{Re} z\,. \tag{1}$$

It is obvious that

$$\widehat{\alpha}_\epsilon(\mathcal{A}) \leq \inf_{z \in \sigma(\mathcal{A})} \operatorname{Re} z\,. \tag{2}$$

We also define in $[-\infty, +\infty[$:

$$\widehat{\omega}_0(\mathcal{A}) = \lim_{t \to +\infty} \frac{1}{t} \log ||e^{-t\mathcal{A}}||\,. \tag{3}$$

Note that there are cases where $\widehat{\omega}_0(\mathcal{A}) = -\infty$ corresponding to semigroups with a faster decay than exponential.

Using Theorem GP1 with $A = -\mathcal{A}$, we get the following statement.

**Theorem GP2: Gearhart-Prüss reformulated**

Let $\mathcal{A}$ be a densely defined closed operator in an Hilbert space $X$ such that $-\mathcal{A}$ generates a contraction semigroup. Then

$$\lim_{\epsilon \to 0} \widehat{\alpha}_\epsilon(\mathcal{A}) = -\widehat{\omega}_0(\mathcal{A})\,. \tag{4}$$

## Remarks.

This version is interesting because it reduces the question of the decay, which is basic in the question of the stability to an analysis of the $\epsilon$-spectra of the operator for $\epsilon$ small, or equivalently the level sets of the subharmonic function $z \mapsto \psi(z) := ||(\mathcal{A} - z)^{-1}||$.

This is not interesting when $\mathcal{A}$ is similar to a selfadjoint operator.

For the application of the previous statements we are interested in $\sup_\nu \psi(\mu + i\nu)$ for $\lambda < \inf\{\mathrm{Re}\, z \,|\, z \in \sigma(\mathcal{A})\}$.

What we should remember:

It is not enough to know the spectrum for determining the decay of the semi-group !

Let us now look at toy models before to consider specific examples in superconductivity and (if time permits) in kinetic theory (Fokker-Planck operator).

TOY MODELS.

# The Airy operator in $\mathbb{R}$

The operator $D_x^2 + ix$ can be defined as the closed extension $\mathcal{A}$ of the differential operator on $C_0^\infty(\mathbb{R})$ :

$$\mathcal{A}_0^+ := D_x^2 + ix\,. \tag{5}$$

$\mathcal{A}$ has compact resolvent, is accretive:

$$\mathrm{Re}\,\langle \mathcal{A}u \,|\, u\rangle \geq 0\,. \tag{6}$$

Hence $-\mathcal{A}$ is the generator of a semi-group $S_t$ of contraction,

$$S_t = \exp -t\mathcal{A}\,. \tag{7}$$

Hence all the results of this theory can be applied.
In particular, we have, for $\mathrm{Re}\,\lambda < 0$

$$||(\mathcal{A} - \lambda)^{-1}|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,. \tag{8}$$

A very special property of this operator is that, for any $a \in \mathbb{R}$,

$$T_a \mathcal{A} = (\mathcal{A} - ia) T_a \,, \tag{9}$$

where $T_a$ is the translation operator :

$$(T_a u)(x) = u(x-a) \,. \tag{10}$$

As immediate consequence, we obtain that the spectrum is empty

$$\sigma(\mathcal{A}) = \emptyset \tag{11}$$

and that the resolvent of $\mathcal{A}$, which is defined for any $\lambda \in \mathbb{C}$ satisfies

$$||(\mathcal{A} - \lambda)^{-1}|| = ||(\mathcal{A} - \mathrm{Re}\,\lambda)^{-1}|| \,. \tag{12}$$

The most interesting property is the control of the resolvent for $\mathrm{Re}\,\lambda \geq 0$.

## Proposition

There exist two positive constants $C_0$ and $C_1$, such that

$$C_1 |\mathrm{Re}\,\lambda|^{-\frac{1}{4}} \exp \frac{4}{3} \mathrm{Re}\,\lambda^{\frac{3}{2}} \leq ||(\mathcal{A} - \lambda)^{-1}|| \leq C_2 |\mathrm{Re}\,\lambda|^{-\frac{1}{4}} \exp \frac{4}{3} \mathrm{Re}\,\lambda^{\frac{3}{2}}\,, \tag{13}$$

(see Martinet [Mart] for this version and Bordeaux-Montrieux–Sjöstrand for further improvments). The proof of the (rather standard) upper bound is based on the direct analysis of the semi-group in the Fourier representation. One can show that $||\exp -t\mathcal{A}||$ decays more than exponentially.

# The Airy complex operator in $\mathbb{R}^+$

Here we mainly describe some results presented in [Alm2], who refers to [IvKol]. We consider the Dirichlet realization $\mathcal{A}^D$ of $D_x^2 + ix$ on the half space. Moreover, by construction, we have

$$\mathrm{Re}\,\langle \mathcal{A}^D u \,|\, u\rangle \geq 0\,,\, \forall u \in D(\mathcal{A}^D)\,. \tag{14}$$

Again we have an operator, which is the generator of a semi-group of contraction. Moreover, the operator has compact inverse, hence the spectrum (if any) is discrete.

Using what is known on the usual Airy operator, Sibuya's theory and a complex rotation, we obtain ([Alm2]) that

$$\sigma(\mathcal{A}^D) = \cup_{j=1}^{+\infty} \{\lambda_j\} \tag{15}$$

with

$$\lambda_j = \exp i\frac{\pi}{3}\, \mu_j \,, \tag{16}$$

the $\mu_j$'s being real zeroes of the Airy function satisfying

$$0 < \mu_1 < \cdots < \mu_j < \mu_{j+1} < \cdots \,. \tag{17}$$

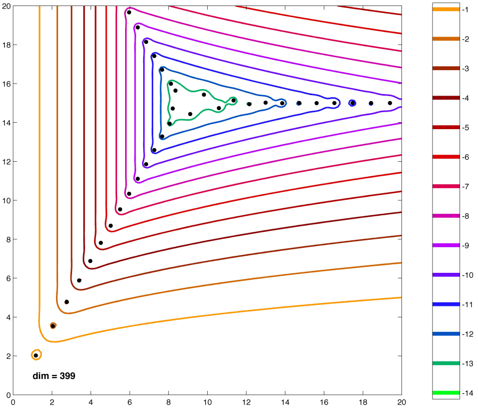
It is also shown in [Alm2] that the vector space generated by the corresponding eigenfunctions is dense in $L^2(\mathbb{R}^+)$.

We arrive now to the analysis of the properties of the semi-group and the estimate of the resolvent.
As before, we have, for $\mathrm{Re}\,\lambda < 0$,

$$||(\mathcal{A}^D - \lambda)^{-1}|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,, \tag{18}$$

If $\mathrm{Im}\,\lambda < 0$ one gets also a similar inequality, so the main remaining question is the analysis of the resolvent in the set $\mathrm{Re}\,\lambda \geq 0\,,\ \mathrm{Im}\,\lambda \geq 0$, which corresponds to the numerical range of the symbol.

Figure: Airy with Dirichlet condition : pseudospectra



Application in superconductivity.

# The model in superconductivity

Consider a superconductor placed in an applied magnetic field and submitted to an electric current through the sample. It is usually said that if the applied magnetic field is sufficiently high, or if the electric current is strong, then the sample is in a normal state. We are interested in analyzing the joint effect of the applied field and the current on the stability of the normal state.

# The model in superconductivity

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To be more precise, let us consider a two-dimensional superconducting sample capturing the entire $xy$ plane. We can assume also that a magnetic field of magnitude $\mathcal{H}^e$ is applied perpendicularly to the sample. Denote the Ginzburg-Landau parameter of the superconductor by $\kappa$ and the normal conductivity of the sample by $\sigma$.

The physical problem is posed in a domain $\Omega$ with specific boundary conditions.
We will only analyze limiting situations where the domains possibly after a blowing argument become the whole space (or the half-space).
In a work in progress [AlmHel] , we analyze the case of bounded domains.
We will mainly work in $2D$ for simplification. $3D$ is also very important.

Then the time-dependent Ginzburg-Landau system (also known as the Gorkov-Eliashberg equations) is in $(0, T) \times \mathbb{R}^2$ :

$$\begin{cases} \partial_t \psi + i\, \kappa\, \Phi \psi = \nabla^2_{\kappa \mathbf{A}} \psi + \kappa^2 \,(1 - |\psi|^2) \psi \,, \\ \kappa^2 \ \text{curl}\ {}^2 \mathbf{A} + \sigma\, (\partial_{\mathbf{t}} \mathbf{A} + \nabla \mathbf{\Phi}) = \kappa \, \mathrm{Im}\, (\bar{\psi} \nabla_{\kappa \mathbf{A}} \psi) + \kappa^{\mathbf{2}} \ \text{curl}\ \mathcal{H}^{\mathbf{e}} \,, \end{cases} \tag{19}$$

where $\psi$ is the order parameter, $\mathbf{A}$ is the magnetic potential, $\Phi$ is the electric potential, and $(\psi, \mathbf{A}, \mathbf{\Phi})$ also satisfies an initial condition at $t = 0$.

# Stationary normal solutions

From (19) we see that if $(0, \mathbf{A}, \mathbf{\Phi})$ is a time-independent normal state solution then $(\mathbf{A}, \mathbf{\Phi})$ satisfies the equality

$$\kappa^2 \text{ curl }^2\mathbf{A} + \sigma\nabla\mathbf{\Phi} = \kappa^{\mathbf{2}} \text{ curl } \mathcal{H}^{\mathbf{e}}\,, \quad \text{div } \mathbf{A} = \mathbf{0} \quad \text{in } \mathbb{R}^{\mathbf{2}}\,. \tag{20}$$

(Note that if one identifies $\mathcal{H}^e$ to a function $h$, then $\text{curl } \mathcal{H}^e = (-\partial_y h\,,\, \partial_x h)$).

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(Note that if one identifies $\mathcal{H}^e$ to a function $h$, then curl $\mathcal{H}^e = (-\partial_y h\,,\,\partial_x h)$).
This could be rewritten as the property that

$$\kappa^2( \text{ curl } \mathbf{A} - \mathcal{H}^\mathbf{e}) + \mathbf{i}\sigma\mathbf{\Phi}\,,$$

is an holomorphic function.
In particular

$$\Delta\Phi = 0 \text{ and } \Delta( \text{ curl } \mathbf{A} - \mathcal{H}^\mathbf{e})) = \mathbf{0}\,.$$

# Results by Almog-Helffer-Pan: $\Phi$ affine

(19) has the following stationary normal state solution

$$\mathbf{A} = \frac{\mathbf{1}}{\mathbf{2J}}(\mathbf{Jx} + \mathbf{h})^2\hat{\mathbf{i}}_\mathbf{y}, \quad \mathbf{\Phi} = \frac{\kappa^2\mathbf{J}}{\sigma}\mathbf{y}\,. \tag{21}$$

Note that

$$\text{curl } \mathbf{A} = (\mathbf{Jx} + \mathbf{h})\,\hat{\mathbf{i}}_\mathbf{z},$$

that is, the induced magnetic field equals the sum of the applied magnetic field $h\hat{i}_z$ and the magnetic field produced by the electric current $Jx\ \hat{i}_z$.

For this normal state solution, the linearization of (19) with respect to the order parameter is

$$\partial_t \psi + \frac{i\kappa^3 Jy}{\sigma}\psi = \Delta\psi - \frac{i\kappa}{J}(Jx+h)^2\partial_y\psi - (\frac{\kappa}{2J})^2(Jx+h)^4\psi + \kappa^2\psi\,. \tag{22}$$

Applying the transformation $x \to x - h/J$ and $\kappa = 1$ for simplification the time-dependent linearized Ginzburg-Landau equation takes the form

$$\frac{\partial\psi}{\partial t} + i\frac{J}{\sigma}y\psi = \Delta\psi - iJx^2\frac{\partial\psi}{\partial y} - \Big(\frac{1}{4}J^2x^4 - 1\Big)\psi\,. \tag{23}$$

Rescaling $x$ and $t$ by applying

$$t \to J^{2/3} t \; ; \; (x,y) \to J^{1/3}(x,y)\,, \tag{24}$$

yields

$$\partial_t u = -(\mathcal{A}_{0,c} - \lambda) u\,, \tag{25}$$

where

$$\mathcal{A}_{0,c} := D_x^2 + (D_y + \frac{1}{2}x^2)^2 + icy\,, \tag{26}$$

and

$$c = 1/\sigma \; ; \; \lambda = J^{-2/3} \; ; \; u(x,y,t) = \psi(J^{-1/3}x, J^{-1/3}y, J^{-2/3}t)\,.$$

Our main problem will be to analyze the long time property of the attached semi-group.
We recall that

$$\mathcal{A}_{0,c} := D_x^2 + (D_y + \frac{1}{2}x^2)^2 + icy \,,$$

**Theorem**

If $c \neq 0$, $\mathcal{A} = \overline{\mathcal{A}_{0,c}}$ has compact resolvent, empty spectrum, and there exist $C, t_0$ such that, for $t \geq t_0$,

$$\| \exp(-t\mathcal{A}) \| \leq \exp\left(-\frac{2\sqrt{2|c|}}{3} t^{3/2} + C t^{3/4}\right). \tag{27}$$

We have also a lower bound using a quasimode construction.

# The case of the half-space

Once the definition of the extended operator $\mathcal{A}_c^+$ has been formulated, we may write

$$\mathcal{A}_c^+ = D_x^2 + (D_y + \frac{1}{2}x^2)^2 + icy \,. \qquad (28)$$

Note that $\mathcal{A}_c^+$ is not self-adjoint. Furthermore, we have that

$$(\mathcal{A}_c^+)^* = \mathcal{A}_{-c}^+ \,.$$

In the present contribution we analyze the spectrum of $\mathcal{A}_c^+$, denoted by $\sigma(\mathcal{A}_c^+)$, and the associated semi-group $\exp -t\mathcal{A}_c^+$.

As in the case of the whole space, we can easily prove that for any $c > 0$, $\mathcal{A}_c^+$ has compact resolvent. Moreover, if $E_0(\omega)$ denotes the ground state energy of the anharmonic oscillator (also called Montgomery operator)

$$\mathcal{M}_\omega := -\frac{d^2}{dx^2} + (\frac{x^2}{2} + \omega)^2,$$

and if

$$E_0^* = \inf_{\omega \in \mathbb{R}} E_0(\omega) = E_0(\omega^*)\,, \tag{29}$$

then

$$\sigma(\mathcal{A}_c^+) \subset \{\lambda \in \mathbb{C}\,,\, \mathrm{Re}\,\lambda \geq E_0^*\}. \tag{30}$$

As mentioned earlier, our interest is in the effect that the Dirichlet boundary condition has on the spectrum $\sigma(\mathcal{A}_c^+)$ and on the semigroups $\exp -t\mathcal{A}_c^+$. Thus, it is interesting to compare them with the analogous entities for the whole-plane problem.

We have seen, that $\sigma(\mathcal{A})$ must be empty and that the decay of the semigroup $\exp(-t\mathcal{A})$ is faster than any exponential rate.
On the other hand, for the half-plane problem we do not expect $\sigma(\mathcal{A}_c^+)$ to be empty. We provide a proof in the asymptotic regimes $c \to +\infty$ and $c \to 0$.
The proof is based on the construction of quasi-modes but one should be aware that it is not immediate to deduce from this construction the existence of an eigenvalue, when we have an approximate eigenvalue.

## Theorem A

There exists $c_0 \geq 0$ such that for $c \geq c_0$

$$\sigma(\mathcal{A}_c^+) \neq \emptyset.$$

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Furthermore, there exists $\mu(c) \in \sigma(\mathcal{A}_c^+)$ which as $c \to +\infty$:

$$\mu(c) \sim c^{2/3} \exp(i\frac{\pi}{3})\alpha_0 + \lambda_1 \exp(-i\frac{\pi}{6})\, c^{-1/3} + \mathcal{O}(c^{-5/6})\,, \quad (31)$$

with $-\alpha_0$ the rightmost zero point of Airy's function, and $\lambda_1$ an eigenvalue of an harmonic oscillator like operator.

$\mu(c)$ is the candidate to be the eigenvalue with smallest real part. One can indeed show that if

$$\mu_m(c) = \inf_{z \in \sigma(\mathcal{A}_c^+)} \operatorname{Re} z \,. \tag{32}$$

then, for all $c > c_0$ $\mu_m(c) \sim \operatorname{Re}\mu(c) + \mathcal{O}(c^{-5/6})$.

The next result is based on Gearhard-Pruss Theorem and is valid for all $c > 0$.

## Theorem B

If $\sigma(\mathcal{A}_c^+) \neq \emptyset$, then

$$\lim_{t \to +\infty} -\frac{\log \| \exp(-t\mathcal{A}_c^+) \|}{t} = \mu_m(c)\,. \quad (33)$$

# The case of $c$ large, Analytic dilation

Instead of dealing with $\sigma(\mathcal{A}_c^+)$, it is more convenient to analyze the spectrum of the operator $\mathcal{P}_\theta$ which is obtained from $\mathcal{A}_c^+$ using a gauge transformation and analytic dilation.
Let $\theta \in \mathbb{C}$. Like for the analysis of resonances, we introduce the dilation operator

$$u \longmapsto (U(\theta)u)(x,y) = e^{-\theta/2}\, u(e^{\theta}x\,,\, e^{-2\theta}y)\,. \tag{34}$$

Set then

$$\mathcal{P}_\theta := U(\theta)^{-1}\mathcal{P}\,U(\theta) = e^{2\theta}\,(D_x - yx)^2 - e^{-4\theta}\,\partial_y^2 + ic\,e^{2\theta}\,y\,, \tag{35}$$

For $\theta = -i\frac{\pi}{12}$, we have

$$\mathcal{P}_{-i\frac{\pi}{12}} = e^{i\pi/3}(D_y^2 + cy) + e^{-i\pi/6}(D_x - xy)^2 \,.$$

Note that $\mathcal{P}_{-i\frac{\pi}{12}}$ is *not* unitarily equivalent to $\mathcal{A}_c^+$. But analytic dilation facilitates the analysis of the spectrum of $\mathcal{A}_c^+$. We introduce the (small) parameter $\epsilon = \frac{1}{c}$ .

After a (real) dilation, we get:

$$\mathcal{B}_\epsilon := \epsilon\big(D_x - xy\big)^2 + i\big(D_y^2 + y\big)\,. \tag{36}$$

The spectrum is unchanged but not the pseudo-spectrum. This can be used to construct quasimodes and the non emptyness of the spectrum.

# Other non self-adjoint problems : The Fokker-Planck operator

If $V$ be a $C^\infty$ potential on $\mathbb{R}^m$, then we consider the operator $K$ defined on $C_0^\infty(\mathbb{R}^{2m})$ by

$$K := -\Delta_v + \frac{1}{4}|v|^2 - \frac{m}{2} + X_0\,, \tag{37}$$

where

$$X_0 := -\nabla V(x)\cdot\partial_v + v\cdot\partial_x \tag{38}$$

$K$ is considered as an unbounded operator on $\mathcal{H} = L^2(\mathbb{R}^{2m})$.

The simplest model of this type is on $\mathbb{R}^2$ ($m = 1$) when we consider the quadratic one $V(x) = \frac{\tilde{\omega}_0^2}{2} x^2$ (with $\tilde{\omega}_0 \neq 0$), for which rather explicit computations can be done:

$$-\frac{\partial^2}{\partial v^2} + \frac{1}{4} v^2 - \frac{1}{2} - \tilde{\omega}_0^2 x \partial_v + v \partial_x \,. \tag{39}$$

In the general case, $X_0$ is the vector field generating the Hamiltonian flow associated with the Hamiltonian: $\mathbb{R}^m \times \mathbb{R}^m \ni (x, v) \mapsto \frac{1}{2}|v|^2 + V(x)$.
We denote its closure by $\overline{K}$ which is now called : the Fokker-Planck operator.

The main result is the following:

**Theorem FP**

For any $V \in C^\infty(\mathbb{R}^m)$, the associated Fokker-Planck operator is maximally accretive.
Let us assume that, for some $\rho_0 > \frac{1}{3}$ and for $|\alpha| = 2$, there exists $C_\alpha > 0$

$$|D_x^\alpha V(x)| \leq C_\alpha (1 + |\nabla V(x)|^2)^{\frac{1-\rho_0}{2}} \,, \qquad (40)$$

and

$$|\nabla V(x)| \to +\infty \,, \text{ as } |x| \to +\infty. \qquad (41)$$

Then $\overline{K}$ has compact resolvent and there exists a constant $C > 0$, such that for all $\nu \in \mathbb{R}$,

$$|\nu|^{\frac{2}{3}} \, ||u||^2 + || \, |\nabla V(x)|^{\frac{2}{3}} \, u \, ||^2 \leq C \, \left( ||(\overline{K} - i\nu)u||^2 + ||u||^2 \right) \, , \forall u \in C_0^\infty \, . \qquad (42)$$

## Corollary FP

Uniformly for $\mu$ in a compact interval, there exists $\nu_0$ and $C$ such that $\mu + i\nu$ is not in the spectrum of $\overline{K}$ and

$$||(\overline{K} - \mu - i\nu)^{-1}|| \leq C\,|\nu|^{-\frac{1}{3}}\,,\ \forall \nu\, s.t.\, |\nu| \geq \nu_0\,. \tag{43}$$

As a main application, we will show:

## Theorem Decay

Under the assumptions of Theorem FP and assuming that $e^{-V} \in L^1(\mathbb{R}^m)$, then there exists $\alpha > 0$ and $C > 0$ so that

$$\forall u \in L^2(\mathbb{R}^{2m}), \quad \left\| e^{-t\overline{K}} u - \Pi_0 u \right\|_{L^2} \leq C e^{-\alpha t} \|u\| \,. \tag{44}$$

where $\Pi_0$ is the projector defined for $u \in L^2(\mathbb{R}^{2m})$ by

$$(\Pi_0 u)(x,v) = \Phi(x,v) \Big( \int \Phi(x,v)\, u(x,v)\, dxdv \Big) / \Big( \int \Phi(x,v)^2\, dxdv \Big) \,, \tag{45}$$

with

$$\Phi(x,v) := \exp -\frac{v^2}{4} \, \exp -\frac{V(x)}{2} \,. \tag{46}$$

# Remarks

References : Hérau-Nier, Villani, Helffer-Nier, Hérau-Hitrick-Sjöstrand,...

There is some PT symmetric aspect.

# Bibliography

Y. Almog.
The motion of vortices in superconductors under the action of electric currents.
Talk in CRM 2008.

Y. Almog.
The stability of the normal state of superconductors in the presence of electric currents.
Siam J. Math. Anal. 40 (2), p. 824-850.

Y. Almog and B. Helffer.
Work in progress.

Y. Almog, B. Helffer, and X. Pan.
Superconductivity near the normal state under the action of electric currents and induced magnetic field in $\mathbb{R}^2$.
To appear in CMP.

Y. Almog, B. Helffer, and X. Pan.

Superconductivity near the normal state under the action of electric currents and induced magnetic field in $\mathbb{R}^2_+$.
To appear in TAMS.

Y. Almog, B. Helffer, and X. Pan.
Superconductivity near the normal state under the action of electric currents and induced magnetic field in $\mathbb{R}^2_+$ II The case $c$ small.
To appear.

P. Bauman, H. Jadallah, and D. Phillips.
Classical solutions to the time-dependent Ginzburg-Landau equations for a bounded superconducting body in a vacuum.
J. Math. Phys., 46 (2005), pp. 095104, 25.

W. Bordeaux Montrieux.
Estimation de résolvante et construction de quasimode près du bord du pseudospectre.
Preprint 2010.

B. Davies

Linear operators and their spectra.
Cambridge studies in advanced mathematics.

B. Davies.
*Wild spectral behaviour of anharmonic oscillators.*
Bull. London. Math. Soc. 32 (2000) 432-438.

Engel and Nagel.
One-parameter semi-groups for linear evolution equations.
Springer Verlag.

S. Fournais and B. Helffer.
Spectral methods in Superconductivity.
Progress in non-linear PDE, Birkhäuser.

B. Helffer.
*Semi-Classical Analysis for the Schrödinger Operator and Applications*,
no. 1336 in Lecture Notes in Mathematics, Springer-Verlag, 1988.

B. Helffer.

The Montgomery model revisited.
To appear in Colloquium Mathematicum.

- B. Helffer and A. Mohamed.
  Sur le spectre essentiel des opérateurs de Schrödinger avec champ magnétique,
  Annales de l'Institut Fourier, tome 38 (1988), fasc.2, p. 95-113.

- B. Helffer and F. Nier.
  Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians.
  Lecture Notes in Mathematics 1862. Springer Verlag.

- B. Helffer and J. Sjöstrand.
  From resolvent bounds to semigroup bounds.
  Appendix of a paper by Sjöstrand. Proceedings of the Evian Conference in 2009.

- R. Henry.

Sur la théorie spectrale de certains problèmes non autoadjoints provenant de la supraconductivité.
Master Thesis.

B.I. Ivlev and N.B. Kopnin.
Electric currents and resistive states in thin superconductors.
Adv. in Physics 33 (1984), p. 47-114.

J. Martinet.
Sur les propriétés spectrales d'opérateurs non-autoadjoints provenant de la mécanique des fluides.
PHD Thesis (December 2009).

J. Sjöstrand.
Resolvent estimates for non-self-adjoint operators via semi-groups.
http://arxiv.org/abs/0906.0094.

L.N. Trefethen.
Pseudospectra of linear operators.
SIAM Rev. 39 (3) (1997), p. 383-406.

L.N. Trefethen and Embree.
A course on Pseudospectra in three volumes (2004).

C. Villani.
Hypocoercivness. Memoirs of the AMS (2009).