Two properties of energy dependent potentials.

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General considerations :

Wave equations with energy dependent potentials are known since the early days of relativistic quantum mechanics. The Pauli-Schrödinger equation is a typical example. Recently they appear in the Hamiltonian formalism of the relativistic many-body problem, i.e. in the manifestly covariant formalism with constraints.

They can be used in the Schrödinger equation to introduce non-linear effect, such that the soliton propagation or interacting clusters.

Studying analytical examples allows us to get acquainted with the effect of the energy dependence of the potential and to show the differences with respect to the usual case.

Purpose of the present work :

To point out two aspects of energy dependent potentials

1 - The regularisation effect in the case of singular potentials

2 - Supersymmetry properties

We consider potentials of the form

$$V(x) = V_0(x) + E\ V_1(x) \tag{1}$$

Analytical cases generally requires $V_1(x) = \gamma V_0(x)$. This work is made in $D = 1$ but it is easily extended to higher dimensions.

Local equivalent potentials.

Much of our arguments are based on the concept of the local equivalent potential.

Consider the Schrödinger equation (we limit the discussion to discrete states)

$$\left[-\frac{d^2}{dx^2} + V_0(x) + E_n V_1(x)\right] \psi_n(x) = E_n \psi_n(x) \ . \tag{2}$$

Suppose the eigenvalue $E_n = \epsilon_n$ to be known, the Schrödinger equation with the local equivalent potential

$$U_n(x) = V_0(x) + \epsilon_n \ V_1(x) \tag{3}$$

yields the same solution for the state $|n>$, with $\psi_n = \phi_n$ but the normalisation is different.

Singular potentials.

In the case of attractive short range potentials, with bound states of negative energy, the energy dependence increases the eigenvalues, in other words it diminishes the binding.

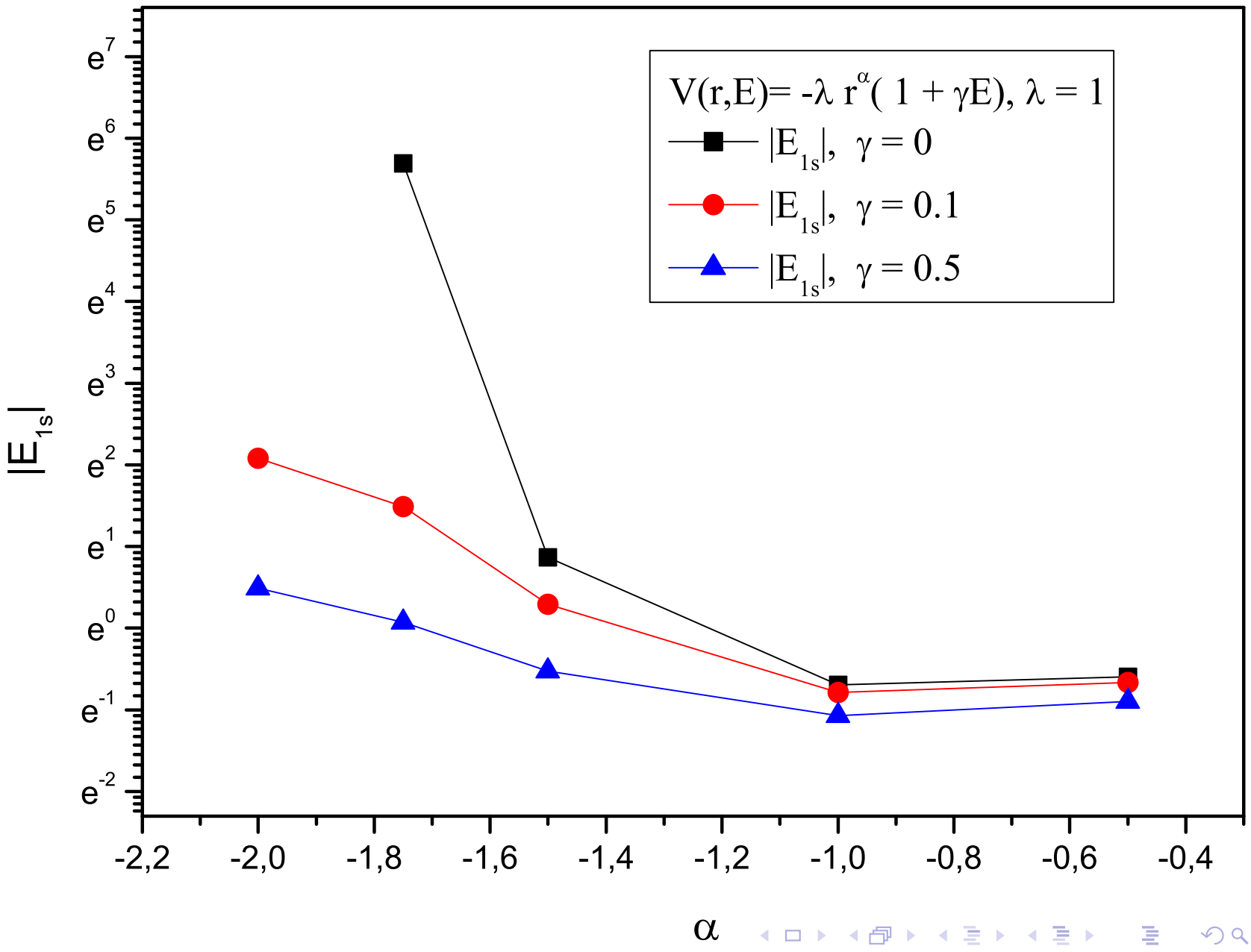
An illustrative example : in $D = 3$ dimensions, take

$$V(r) = -\lambda \; r^{\alpha} \; [1 + \gamma E_n] \; , \tag{4}$$

with $\lambda > 0$, $\alpha < 0$ and $\gamma \geq 0$.

NOTA : For this form of potentials, γ has to be positive to ensure the positivity of the density distribution.

Let us look at some numerical estimates.


V(r,E)= -λ r^α(1 + γE), λ = 1
|E1s|, γ = 0
|E1s|, γ = 0.1
|E1s|, γ = 0.5
|E1s|
e7
e6
e5
e4
e3
e2
e1
e0
e-1
e-2
-2,2
-2,0
-1,8
-1,6
-1,4
-1,2
-1,0
-0,8
-0,6
-0,4
α

These numerical results are no more than an indication, specially for $\alpha \to -2$.

Theorem :

Let $V(x) = -\lambda f(|x|)[1 + \gamma E]$

$\lambda > 0$; $f(|x|) \to 0$ as $x \to \pm\infty$, possibly singular in $x = 0$,

If $V(x)$ admits a bound state solution $\psi_\epsilon(x)$ (square integrable !) with $E = -\epsilon$,

$$H = -\frac{d^2}{dx^2} + V(x) \tag{5}$$

is bounded from below.

As stated above, $\psi_\epsilon(x)$ is also solution of the local equivalent potential

$$\left[-\frac{d^2}{dx^2} - U_\epsilon(|x|)\right]\phi_\epsilon(x) = -\epsilon\phi_\epsilon(x) \tag{6}$$

with

$$U_\epsilon(|x|) = \lambda f(|x|)[1-\gamma\epsilon] \ . \tag{7}$$

We multiply from the left with $\phi_\epsilon(x)$ and integrate. It yields

$$-\int_{-\infty}^{\infty}\phi_\epsilon(x)\frac{d^2}{dx^2}\phi_\epsilon(x)dx - \int_{-\infty}^{\infty}|\phi_\epsilon(x)|^2 U_\epsilon(|x|)dx = -\epsilon \ . \tag{8}$$

The first contribution on the left is positive definite (kinetic energy), thus, with $\lambda > 0$, $\gamma > 0$ and $f(|x|) > 0$, this equation is satisfied if and only if

$1-\gamma\epsilon > 0$, namely $\epsilon < 1/\gamma$.

quod erat demonstrandum !

This result applies to a large class of potentials, including singular cases.

First example : The D = 1 Hydrogen atom

L.K.Haines and D.R. Roberts Am. J. of Physics **37** (1969) 1145.

We use $\hbar = 2m = 1$

$$[-\frac{d^2}{dx^2} - \frac{e^2}{|x|}(1+\gamma E)]\psi(x) = E\psi(x) \tag{9}$$

Setting $E = -\frac{1}{a_0^2\alpha^2}$, $e^2 = \frac{2}{a_0}$ (a_0 is the Bohr orbit) and $x = \frac{1}{2}a_0\alpha z$ yields

$$\left[\frac{d^2}{dz^2} - \frac{1}{4} + \frac{1}{|z|}(\alpha - \frac{\gamma'}{\alpha})\right]\psi(z) = 0 \tag{10}$$

which is the Whittaker's equation (here $\gamma' = \gamma/a_0^2$).

The solution is given by

$$\psi(z) = e^{-z/2}\ z\ U(1-\kappa, 2, z)\ , \tag{11}$$

where $\kappa = \alpha - \gamma'/\alpha$

The behaviour as $z \to 0$ is given by

$$\psi(z) \to \frac{C_\pm}{\Gamma(-\kappa)}[-\frac{1}{\kappa} + |z| \ln |z| + O(z)\ . \tag{12}$$

The matching condition at $z = 0$ yields

$$-\frac{1}{\kappa}\frac{C_+}{\Gamma(-\kappa)} = -\frac{1}{\kappa}\frac{C_-}{\Gamma(-\kappa)}\ . \tag{13}$$

Ground state : this is satisfied as soon as $\kappa \neq 0$, which implies

$$\alpha - \gamma'/\alpha > 0 \quad \text{i.e.} \quad \alpha > 1/\gamma'\ . \tag{14}$$

Second example : The D = 3 square well potential.

$V(r) = -\lambda(1 + \gamma E)\Theta(R - r)$

It simulates a singular potential when $\lambda \to \infty$. The solutions are well known. For the 1s state, if

$$k^2 = \lambda(1 + \gamma E_{1s}) + E_{1s} \text{ and } \kappa^2 = -E_{1s} \ , \tag{15}$$

the matching of the logarithmic derivatives at $r = R$ yields

$$\tan(kR) = -k/\kappa \ . \tag{16}$$

The existence of a bound state requires $k^2 > 0$, which yields immediately

$$E_{1s} > -\frac{\lambda}{1 + \gamma \ \lambda} \ . \tag{17}$$

The matching conditions can be solved numerically to the desired degree of accuracy.

The numerical estimate for the $1s$ and the $1p$ states yields

λ	E_{1s}	E_{1p}
5	-0.4033	unbound
10	-0.9827	-0.0168
50	-1.7491	-1.4966
100	-1.871	-1.7409
500	-1.9736	-1.9469
1000	-1.9868	-1.9734

Table: HERE, $R = 1$ AND $\gamma = .5$

The Pöschl-Teller case

For the ground state

$$\left[-\frac{d^2}{dx^x}-\frac{\lambda(1+\gamma E_0)}{\cosh x^2}\right]\psi_0(x)=E_0\psi_0(x) \tag{18}$$

as $\lambda\to\infty$

$$E_0\to-\frac{1}{1/\lambda+\gamma}\ . \tag{19}$$

Supersymmetry.

We propose a way to extend the supersymmetry algebra to energy dependent potentials, by generalising the superpotential. It uses the concept of local equivalent potential.

We recall that the superpotential is defined by ($D = 1$ - dimension)

$$W(x) = -\frac{\phi_0'(x)}{\phi_0(x)} \ . \tag{20}$$

Let $\{\psi_n(E_n, x), E_n\}$ be the set of eigenfunctions and eigevalues of $V_0(x)(1 + \gamma E_n)$
To each state $\psi_k(E_k, x) \in \{\psi_n(E_n, x)\}$, a potential $U_k(x)$ can be associated in such a way that

$$\left[-\frac{d^2}{dx^2} + U_k(x)\right] \psi_k(E_k, x) = E_k \psi_k(E_k, x). \tag{21}$$

Obviously we have

$$U_k(x) = V_0(x)(1 + \gamma E_k) = \lambda_k V_0(x). \tag{22}$$

$U_k(x)$ local, energy independent

$$\left[-\frac{d^2}{dx^2} + U_k(x)\right] \phi_{k,m}(x) = \varepsilon_{k,m}\phi_{k,m}(x) \quad m = 0, 1, 2, \ldots \tag{23}$$

generates a set $\{\phi_{k,m}(x), \varepsilon_{k,m}\}$ of eigenfunctions and eigenvalues with

$$\phi_{k,k}(x) = \psi_k(E_k, x) \qquad ; \qquad \varepsilon_{k,k} = E_k \tag{24}$$

and

$$\phi_{k,m}(x) \neq \psi_k(E_k, x); \qquad \varepsilon_{k,m} \neq E_k \qquad \text{if} \qquad m \neq k. \tag{25}$$

$U_k(x)$ differs from $V_0(x)$ only through its coupling constant.
A corresponding superpotential can be defined from the ground state wave function of $U_k(x)$ $\phi_{k,0}(x)$:

$$W_k(E_k, x) = -\frac{\phi'_{k,0}(x)}{\phi_{k,0}(x)}, \tag{26}$$

Each eigenstate of a potential depending on energy through its coupling constant is simultaneously an eigenstate of a local energy independent potential. A superpotential can be associated to this state. The ensemble of these superpotentials provides the generalisation of the superpotential.

Orthogonality relations :
For $\{k\}$ of energy E_k, $\psi_k(E_k, x) \equiv \phi_{k,k}(x)$ up to the normalisation factor. Then, the orthogonality condition with respect to the $\phi_{k,m}(x)$ functional space is ensured by

$$\int \psi_k(E_k, x)\phi_{k,m}(x)dx = \delta_{km}. \tag{27}$$

The orthogonality of $\phi_{k,k}(x)$ with respect to the $\psi_n(E_n, x)$ is given by

$$\int \psi_n(E_n, x)\phi_{k,k}(x)\left[1 - \frac{\partial V(E_n, x)}{\partial E_n}\right] dx = \delta_{kn}. \tag{28}$$

The generalised superpotential allows us to generalise the basic differential operators :

$$A_n^+ = -\frac{d}{dx} + W_n(E_n, x) \qquad ; \qquad A_n^- = \frac{d}{dx} + W_n(E_n, x). \quad (29)$$

They obey

$$\left[A_n^+, A_n^-\right] = 2W_n'(E_n, x) \quad (30)$$

and

$$A_n^+ A_n^- = \left[-\frac{d^2}{dx^2} + W_n^2(E_n, x) - W_n'(E_n, x)\right] = H_n - \varepsilon_{n,0} \ . \quad (31)$$

$\varepsilon_{n,0}$ is defined from (21) for the ground state $m = 0$.

The supersymmetric partners H_n^- and H_n^+ are shifted in energy with respect to the spectrum of the reference H_n, in such a way that $E_0^- = 0$.
With the superpotential depending on the state, the shift is different for each state.

$$H_n^- = A_n^+ A_n^- + (\varepsilon_{n,0} - \varepsilon_{0,0}) \qquad n = 0, 1, 2, 3, \ldots . \tag{32}$$

Moreover, the parameters of H_n^+ has to be shifted by one unit with respect to those of H_n^- to ensure $\psi_n^+(E_{n+1}, x)$ and $\psi_{n+1}^-(E_{n+1}, x)$.

$$H_n^+ = A_{n+1}^- A_{n+1}^+ + (\varepsilon_{n+1,0} - \varepsilon_{0,0}) \qquad n = 0, 1, 2, 3, \ldots . \tag{33}$$

These two Hamiltonians satisfy

$$H_n^- \psi_n^-(E_n, x) = E_n^- \psi_n^-(E_n, x) \;\; ; \;\; H_n^+ \psi_n^+(E_{n+1}, x) = E_n^+ \psi_n^+(E_{n+1}, x) \, . \tag{34}$$

With these modifications taken into account, the basic rules of supersymmetry are fulfilled by energy dependent potential :
- factorisation of the Hamiltonians
- relationships between eigenvalues and wave functions of H_n^- and H_n^+.

$$E_n^+ = E_{n+1}^- \, . \tag{35}$$

$$\psi_n^+(E_{n+1}, x) = A_{n+1}^- \psi_{n+1}^-(E_{n+1}, x) \tag{36}$$

$$\psi_{n+1}^-(E_{n+1}, x) = A_n^+ \psi_n^+(E_{n+1}, x). \tag{37}$$

The harmonic oscillator

$$V_0(x) = \omega^2 x^2.$$

$$H_n\psi_n(E_n, x) = \left[-\frac{d^2}{dx^2} + \omega^2(1+\gamma E_n)x^2\right]\psi_n(E_n, x) = E_n\psi_n(E_n, x) \tag{38}$$

$$\psi_n(E_n, x) = \mathcal{H}_n(\sqrt{\alpha_n}x)e^{-\alpha_n x^2/2}, \qquad n = 0, 1, 2, ..., \tag{39}$$

where $\mathcal{H}_n$ are the Hermite polynomials. The eigenvalues and the coupling constant are linked by

$$\alpha_n^2 = \omega^2(1+\gamma E_n); \qquad E_n = (2n+1)\alpha_n \qquad \gamma < 0. \tag{40}$$

$$E_n = \frac{1}{2}\left[\omega^2\gamma(2n+1)^2 \pm \omega(2n+1)\sqrt{\omega^2\gamma^2(2n+1)^2 + 4}\right] \ . \tag{41}$$

Only the roots with the positive square root lead to normalisable wave functions.

The problem is governed by the parameter α_n, and the superpotential is given by

$$W_n(E_n, x) = \alpha_n x. \tag{42}$$

It yields

$$H_n^- \psi_n^-(\alpha_n, x) = \left[-\frac{d^2}{dx^2} + \alpha_n^2 x^2 - \alpha_0\right] \psi_n^-(\alpha_n, x) = E_n^- \psi_n^-(\alpha_n, x). \tag{43}$$

$$E_n^- = E_n - \alpha_0 = (2n+1)\alpha_n - \alpha_0. \tag{44}$$

$$\begin{aligned} H_n^+ \psi_n^+(\alpha_{n+1}, x) &= \left[-\frac{d^2}{dx^2} + 2\alpha_{n+1} + \alpha_{n+1}^2 x^2 - \alpha_0\right] \psi_n^+(\alpha_{n+1}, x) \\ &= E_n^+ \psi_n^+(\alpha_{n+1}, x). \end{aligned} \tag{45}$$

It is easy to check that the all relationships are verified. In particular

$$E_n^+ = (2n+3)\alpha_{n+1} - \alpha_0 = E_{n+1}^- \ . \tag{46}$$

Conclusions.

The present work is devoted to the energy dependent potentials. We consider a linear dependence, which leads to a coherent theory.

For attractive potentials :

The energy dependence affects essentially the lowest states, which are less bound. It has a regularisation effect in the sense that the corresponding Hamiltonians are bounded from below, leading to finite binding energies in the case of singular potentials.

Supersymmetry :

We propose a construction extending the concept of superpotential to the energy dependent case. With this generalisation taken into account, the rules of supersymmetry are applicable.