Pseudo-Hermitian Interactions in the Dirac Equation

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Outline

In recent years massless Dirac equation in $(2+1)$ dimensions has drawn a lot of attention primarily because of its similarity to the equation governing the motion of charge carriers in graphene.
K. S. Novoselov et al. Science 306 (2004), 666,
K. S. Novoselov et al. Nature 438 (2995) 197
In view of the fact that electrostatic fields alone can not provide confinement of the electrons, there have been quite a number of works on exact solutions of the relevant Dirac equation with different magnetic field configurations:

- square well magnetic barrier [De Martino et al. PRL 98 (2007) 066802]
- non zero magnetic fields in dots
  [D. Wang and G. Jin Phys. Lett. A 373 (2009) 4082]
- decaying magnetic fields
  [T. K. Gosh, J. Phys: Cond. Mat. 21 (2009) 045505]
- solvable magnetic field configurations
  [S. Kuru, J. M. Negro, and L. M. Nieto, J. Phys.: Cond. Mat. 21 (2009) 455305
  E. Milpas, M. Torres and G. Murguia, J. Phys: Cond. Mat. **23** (2011) 245304.]

On the other hand, at the same time there have been some investigations into the possible role of non Hermiticity and $\mathcal{PT}$ symmetry in:

- graphene
  [M. Fagotti, C. Bonati, D. Logoteta, P. Marconcini and M. Macucci, Phys.Rev **B83** (2011) 241406
  A. Szameit, M.C. Rechtsman, O. Bahat-Treidel and M. Segev, Phys.Rev **A84** (2011) 021806(R)
  K. Esaki, M. Sato, K. Hasebe and M. Kohmoto, Phys.Rev **B84** (2011) 205128]
- optical analogues of relativistic quantum mechanics
  [S. Longhi, App.Phys.Lett **B104** (2011) 453]
- relativistic non Hermitian quantum mechanics
  [S. Longhi, Phys.Rev.Lett **105** (2010) 013903]
- photonic honeycomb lattice
  [H. Ramezani, T. Kottos, V. Kovanis and D.N. Christodoulides, Phys.Rev **A85** (2012) 013818]

- Here our objective is to widen the scope of incorporating non Hermitian interactions in $(2+1)$ dimensional Dirac equation. We shall introduce non $\mathcal{PT}$ symmetric but non Hermitian interactions by using imaginary vector potentials.
- It may be noted that imaginary vector potentials have been studied previously in connection with the localization problem:
  [N. Hatano and D. Nelson, Phys.Rev.Lett, **77** (1996) 570,
  J. Feinberg and A. Zee, Phys.Rev, **E59** (1999) 6433]
- To be more specific, we shall consider η-pseudo Hermitian interactions within the framework of $(2+1)$ dimensional Dirac equation
  Rasbha and scalar interaction with imaginary couplings (1+1) Dirac eq. :
  B.P. Mandal and S. Gupta, Mod.Phys.Lett **A25** (2010) 1723.
- In particular, we shall examine exact bound state solutions in the presence of imaginary magnetic fields arising out of imaginary vector potentials.
  We shall also obtain the metric and it will be shown that the resulting Dirac Hamiltonians are η-pseudo Hermitian.

Massless Dirac equation in (2+1) dimensions in an external potential A

$$H\psi = E\psi, \qquad H = v_F \boldsymbol{\sigma}.P = v_F \begin{pmatrix} 0 & P_- \\ P_+ & 0 \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

where $P_\pm = (P_x \pm iP_y) = (p_x + eA_x) \pm i(p_y + eA_y)$ and the two components indicate spin up and spin down states. Next, squaring:

$$H^2\psi = v_F^2 \begin{pmatrix} P_-P_+ & 0 \\ 0 & P_+P_- \end{pmatrix} \psi = E^2\psi$$

Let us now consider the vector potential to be

$$A_x = 0, \qquad A_y = f(x)$$

so that the magnetic field is given by

$$\mathcal{B}_z(x) = f'(x)$$

For the above choice of vector potentials the component wave functions can be taken of the form

$$\psi_{1,2}(x,y) = e^{ik_y y}\phi_{1,2}(x)$$

The equations for the ϕ components are found to be

$$\left[-\frac{d^2}{dx^2} + W^2(x) - W'(x)\right]\phi_1(x) = \epsilon^2\phi_1(x)$$

$$\left[-\frac{d^2}{dx^2} + W^2(x) + W'(x)\right]\phi_2(x) = \epsilon^2\phi_2(x)$$

where $\epsilon = (E/\hbar v_F)$ and the function $W(x)$ is given by

$$W(x) = k_y + f(x)$$

where $f(x) = A_y(x)$

Example n.1 – The Complex Decaying Magnetic Field

It is now necessary to choose the function $f(x)$. Our first choice for this function is

$$f(x) = (A + iB)\ e^{-x}, \qquad A, B > 0$$

and it leads to a complex exponentially decaying magnetic field

$$\mathcal{B}_z(x) = -(A + iB)e^{-x}$$

For $B = 0$ or a purely imaginary number the magnetic field is an exponentially decreasing one and we recover the case considered by Ghosh and Kuru et al. The equation of the component ϕ_1 can be cast as:

$$\left[-\frac{d^2}{dx^2} + V_1(x)\right]\phi_1 = \left(\epsilon^2 - k_y^2\right)\phi_1$$

where

$$V_1(x) = (A + iB)^2\ e^{-2x} - (2k_y + 1)(A + iB)\ e^{-x}$$

It is not difficult to recognize $V_1(x)$ as the (complex) Morse potential whose solutions are well known [S. Flugge, Practical Quantum Mechanics; F. Cooper, A. Khare and U. Sukhatme, Supersymmetry in Quantum Mechanics]

Using these results we find

$$E_{1,n} = \pm\hbar v_F \sqrt{k_y^2 - (k_y - n)^2},$$
$$\phi_{1,n} = y^{k_y - n} e^{-y/2} L_n^{(2k_y - 2n)}(y), \qquad n = 0, 1, 2, ... < [k_y]$$

where $y = 2(A + iB)e^{-x}$ and $L_n^{(a)}(y)$ denotes generalized Laguerre polynomials.

Note that for the energy levels to be real it follows from that the corresponding eigenfunctions are acceptable when the condition $k_y \geq 0$ holds. For $k_y < 0$, the wave functions are not normalizable.

Let us now examine the spin down component ϕ_2.
Since ϕ_1 is known one can always use the intertwining relations to obtain ϕ_2:

$$v_F p_+ \psi_{1=} E \psi_2$$

Nevertheless, for the sake completeness we present the detailed results for ϕ_2. In this case the potential analogous reads

$$V_2(x) = (A + iB)^2 \; e^{-2x} - (2k_y - 1)(A + iB) \; e^{-x}$$

Clearly, $V_2(x)$ can be obtained from $V_1(x)$ by the replacement $k_y \to k_y - 1$ and so the solutions can be obtained from as

$$\begin{aligned} E_{2,n} &= \pm \hbar v_F \sqrt{k_y^2 - (k_y - n - 1)^2}, \\ \phi_{2,n} &= y^{k_y - n - 1} e^{-y/2} L_n^{(2k_y - 2n - 2)}(y), \qquad n = 1, 2, ... < [k_y - 1] \end{aligned}$$

Note that in this case the $n = 0$ state is missing from the spectrum so that it is a spin up singlet state.

Furthermore,

$$E_{1,n+1} = E_{2,n}$$

so that, while the ground state is a singlet, the excited are doubly degenerate. Similarly the negative energy states are also paired.

The precise structure of the wave functions are as follows (we present only the positive energy solutions):

$$E_0 = 0, \quad \psi_0 = \begin{pmatrix} \phi_{1,0} \\ 0 \end{pmatrix}$$
$$E_n = \hbar v_F \sqrt{k_y^2 - (k_y - n)^2}, \quad \psi_n = \begin{pmatrix} \phi_{1,n} \\ \phi_{2,n} \end{pmatrix}, \qquad n = 1, 2, ... < [k_y - 1]$$

- It may also be noted that the dispersion relation is no longer linear as it should be in the presence of interactions.
- So, we find that it is indeed possible to create bound states with an imaginary vector potential.
- We shall now demonstrate the above results for a second example.

Example n.2: Complex hyperbolic magnetic field

Here we choose for $f(x)$ a form which leads to an effective potential of the complex hyperbolic Rosen-Morse type :

$$f(x) = A \tanh(x - i\alpha), \qquad A \text{ and } \alpha \text{ real}$$

In this case the complex magnetic field is given by

$$\mathcal{B}_z(x) = A\,\text{sech}^2(x - i\alpha)$$

Note that for $\alpha = 0$ we get back the results of Kuru et al. and Milpas et al..
Using the second order formalism we find for the upper component ϕ_1:

$$\left[-\frac{d^2}{dx^2} + U_1(x)\right]\phi_1 = (\epsilon^2 - k_y^2 - A^2)\phi_1$$

where

$$U_1(x) = k_y^2 - A(A+1)\,\text{sech}^2(x - i\alpha) + 2Ak_y \tanh(x - i\alpha)$$

This is an hyperbolic Rosen Morse potential with known energy values and eigenfunctions. In our case the solutions are:

$$E_{1,n} = \pm\hbar v_F \sqrt{A^2 + k_y^2 - (A-n)^2 - \frac{A^2 k_y^2}{(A-n)^2}}$$

$$\phi_{1,n}(x) = (1-y)^{s_1/2}(1+y)^{s_2/2} P_n^{(s_1,s_2)}(y), \qquad n = 0,1,2,.... < [A - \sqrt{Ak_y}]$$

where

$$y = \tanh\, x, \quad s_{1,2} = A - n \pm \frac{Ak_y}{A-n}$$

and $P_n^{(a,b)}(x)$ are Jacobi polynomials.
The energy values corresponding to the lower component of the spinor, namely, ϕ_2 are found by replacing A by $(A-1)$ and ϕ_2 can be found out using the intertwining relation .

$\mathcal{PT}$ symmetry: ($\mathcal{P}$) parity inversion and ($\mathcal{T}$) time inversion
[C.M. Bender and S. Boettcher PRL 80 (1998) 5243]

$$\mathcal{P}x\mathcal{P} = -x, \quad \mathcal{P}p\mathcal{P} = -p$$

$$\mathcal{T}x\mathcal{T} = x, \quad \mathcal{T}p\mathcal{T} = -p, \quad \mathcal{T}i\mathcal{T} = -i$$

A non Hermitian Hamiltonian $H \neq H^\dagger$ is $\mathcal{PT}$ symmetric if $H\mathcal{PT} = \mathcal{PT}H$
This last relation leads to the condition

$$V(x) = V^*(-x)$$

Neither of the potentials $V_{1,2}(x)$ or $U_{1,2}$ discussed above satisfy the condition and therefore, are not $\mathcal{PT}$ symmetric.

Nevertheless they admit real eigenvalues!

An Hamiltonian H is said to be η-pseudo Hermitian if there exist an hermitian operator such that

$$\eta H \eta^{-1} = H^{\dagger}$$

- The eigenvalues of a η-pseudo Hermitian Hamiltonian are either all real or are complex conjugate pairs.
  [A. Mostafazadeh, J. Math. Phys. A 43 (2002) 1418246].
- In view of the fact that in the present examples the eigenvalues are all real, one is tempted to conclude that the interactions are η pseudo Hermitian.

To show this consider example n. 1 and the Hermitian operator
[Z. Amhed, Phys. Lett. A 290 (2001), 19.]

$$\eta = e^{-\theta p_x}, \quad \theta = \arctan \frac{B}{A}$$

$$\Rightarrow \quad \eta c \eta^{-1} = c, \quad \eta p_x \eta^{-1} = p_x, \quad \eta V(x) \eta^{-1} = V(x + i\theta)$$

We recall that in both the cases considered here the Hamiltonian is of the form

$$H = v_F \boldsymbol{\sigma}.P = v_F \begin{pmatrix} 0 & P_- \\ P_+ & 0 \end{pmatrix}$$

where for example n. 1

$$P_\pm = p_x \pm i p_y \pm i(A + iB) e^{-x}$$

Then:

$$H^\dagger = v_F \begin{pmatrix} 0 & P_+^\dagger \\ P_-^\dagger & 0 \end{pmatrix}$$

Now from

$$P_+^\dagger = p_x - i p_y - i(A - iB) e^{-x}, \quad P_-^\dagger = p_x + i p_y + i(A - iB) e^{-x}$$

it follows that:

$$\eta P_+ \eta^{-1} = p_x + i p_y + i(A - iB) e^{-x} = P_-^\dagger, \quad \eta P_- \eta^{-1} = p_x - i p_y - i(A - iB) e^{-x} = P_+^\dagger$$

- To demonstrate pseudo Hermiticity of the Dirac Hamiltonian, let us consider the operator $\eta' = \eta.\mathcal{I}_2$ where $\mathcal{I}_2$ is the (2×2) unit matrix. Then it follows from the above:

$$\eta' H {\eta'}^{-1} = H^\dagger$$

- Thus the full Dirac Hamiltonian with a complex decaying magnetic field (example n. 1) is η-pseudo Hermitian.
- For the magnetic field given of example n. 2 the metric operator η can be found by a similar straightforward calculation:

$$\eta = e^{-2\alpha p_x}$$

  and also in this case the full Dirac Hamiltonian is found is η-pseudo Hermitian.
- Other examples could be given

Conclusions

- we have studied $(2+1)$ dimensional massless Dirac equation in the presence of complex magnetic fields
- It has been shown that such complex magnetic vector potential can create bound states.
- In the case of the complex decaying and hyperbolic magnetic field the spectrum is real.
- It has also been shown that the Dirac Hamiltonians in the presence of such magnetic fields are η-pseudo Hermitian.

Outlook

- Study generation of bound states using other types of magnetic fields e.g, periodic magnetic fields.