From Krein to S-spaces: a short step

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An ad hoc *example*

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. Multiplication by V is selfadjoint with respect to the new inner product $[\,\cdot\,, -]$, whatever it means.

Details later

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An inner product space $\mathscr{E}$ is said to be a *S–space* if there is a Hilbert space structure in $\mathscr{E}$ with the positive definite inner product $\mathscr{E} \times \mathscr{E} \ni (f, g) \mapsto \langle f, g \rangle \in \mathbb{C}$ and a unitary operator U in the Hilbert space $(\mathscr{E}, \langle \cdot\, , - \rangle)$ such that

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$$[f, g] = \langle Uf, g \rangle, \quad f, g \in \mathscr{E};$$

The latter is not uniquely determined though its role is more than auxiliary. We refer to $(\mathscr{E}, \langle \,\cdot\, , - \rangle, U)$ as a *Hilbert space realization* of the S–space in question.

Next-door consequences

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2^{o} There is a unique (independent of a particular choice of a Hilbert space realization) topology in $\mathscr{E}$ which makes the S–inner product separately continuous.
Consequently, closedness, closure, core and continuity (hence boundedness) of an operator are uniquely designated.

The conjugate of an S–space

If $(\mathscr{E}, [\,\cdot\,, -])$ is an S–space then so is $(\mathscr{E}, [\,\cdot\, -])$ with

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call the latter the *conjugate* of the former. Moreover, if $(\mathscr{E}, \langle\,\cdot\,,-\rangle, U)$ is a Hilbert space realization of $(\mathscr{E}, [\,\cdot\,,-])$ then $(\mathscr{E}, \langle\,\cdot\,,-\rangle, U^*)$ is a Hilbert space realization of $(\mathscr{E}, [\,\cdot\,,-]_{\mathrm{con}})$.

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Riesz-like

For $\Phi \in \mathscr{E}'$ there is a uniquely determined pair (g_1, g_2) of vectors of $\mathscr{E}$ such that

$$\Phi(f) = [f, g_1] = [f, g_2]_{\mathrm{con}}, \quad f \in \mathscr{E}.$$

Consequently, there is a unique linear bicontinuous operator D on $\mathscr{E}$ such that

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Suppose $(\mathscr{E},\langle\,\cdot\,,-\rangle,U)$ is any Hilbert space realization of the S–space $\mathscr{E}$. Then $D=(U^*)^2$.

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1. the dissymmetry operator D is unitary in $(\mathscr{E},\langle\,\cdot\,,-\rangle,U)$,
2. the operator U^2 is independent of a particular choice of a Hilbert space realization.

Given a densely defined operator A in $\mathscr{E}$, a densely defined operator $A^\natural$ is said to be a (*right*) *adjoint* of A and another densely defined operator ${}^\natural A$ (*left*) *adjoint* of A if

$$[Af, g] = fA^\natural g, \quad f \in \mathscr{D}(A),\, g \in \mathscr{D}(A^\natural);$$
$$[f, Ag] = [{}^\natural Af, g], \quad f \in \mathscr{D}({}^\natural A),\, g \in \mathscr{D}(A).$$

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$A^\natural = {}^\natural\! A$ if and only if $DA^* = A^*D$. In addition to this, $D^\natural = {}^\natural\! D = D^{-1} = D^*$.

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Consequently, $A^{\natural\natural} = D^{\natural}\overline{A}\,D$ and ${}^{\natural\natural}\!A = D\overline{A}\,D^{\natural}$.

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A is right symmetric if and only if it is left symmetric. A is right selfadjoint if and only if it is left selfadjoint.
Furthermore, A is S–symmetric if and only if $A \subset UA^*U^*$ and it is S–selfadjoint if $A = UA^*U^*$ holds.

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Another example

Consider the two-sided ℓ^2 space and a two-sided weighted shift, that is

$$S = U^* D$$

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Aplication

q-deformed version of the quantum harmonic oscillator.