On the spectrum of a bent chain graph

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Outline:

1. which graph I mean
2. one-ring graph
3. more-than-one-ring graph

Quantum graphs

Quantum graphs: Schrödinger operators with graph configuration spaces

Aim: to study spectral properties of a Schrödinger operator supported by a graph

Geometry of such a graph:

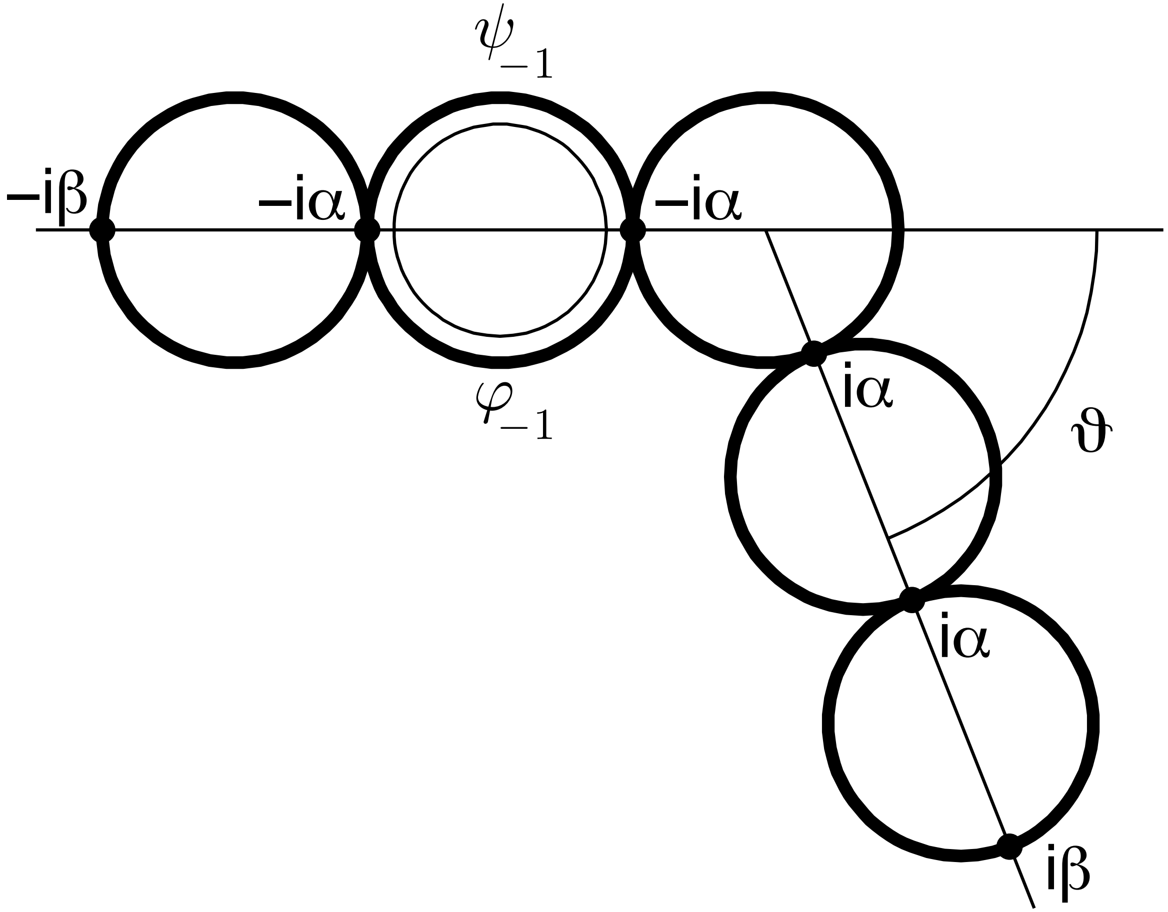
- enters through the adjacency matrix of the graph and the lengths of its edges
- graph is a subset of $\mathbb{R}^n$ with the geometry inherited from the ambient space

Generally, we know that perturbing geometry of graphs (deformation) can change the spectrum ($eg.$ it may give rise to eigenvalues)

Particle is nonrelativistic and spinless living on the graph.

Models (almost) solvable have a prominent position.

The Model



Our model consists of identical rings of radius (wlog) one connected at the touching points by a point interaction.

The particle Hamiltonian is the (negative) Laplacian acting as $\psi \to -\psi''$.

The Hilbert space for the graph is the orthogonal sum of $2n+1$ L^2 spaces referring to the graph links; its elements will be denoted as

$$(\varphi_{-n}, \psi_{-n}, \dots, \varphi_{-1}, \psi_{-1}, \varphi_0, \psi_0, \varphi_1, \psi_1, \dots, \varphi_n, \psi_n)$$

with the coordinates at the loop taken anticlockwise.

The eigenfunction with energy $E = k^2 \neq 0$ is a linear combination of the functions $e^{\pm ikx}$. We parametrize the wavefuction components on the jth ring as $(j = -n, \dots, 0, \dots, n)$

$$\varphi_j(x) = A_j^+ e^{ikx} + A_j^- e^{-ikx}$$
$$\psi_j(x) = B_j^+ e^{ikx} + B_j^- e^{-ikx}$$

The point interaction (δ*–coupling*) in the contact point

$$\varphi_{j-1}(\pi) = \psi_{j-1}(0) = \psi_j(\pi) = \varphi_j(0)$$
$$\psi'_{j-1}(0) - \psi'_j(\pi) - \varphi'_{j-1}(\pi) + \varphi'_j(0) = \pm i\alpha\psi_j(0).$$

At the endpoints we require

$$\varphi_{-n}(0) = \psi_{-n}(\pi) \qquad \varphi_n(\pi) = \psi_n(0)$$
$$\varphi'_{-n}(0) - \psi'_{-n}(\pi) = -i\beta\psi_{-n}(\pi) \qquad \psi'_n(0) - \varphi'_n(\pi) = i\beta\psi_n(0).$$

The bending angle ϑ is supposed to take values from $[0, \pi]$, regardless of the fact that for $\vartheta \geq 2\pi/3$ it is not possible to consider the graph embedded in the plane.

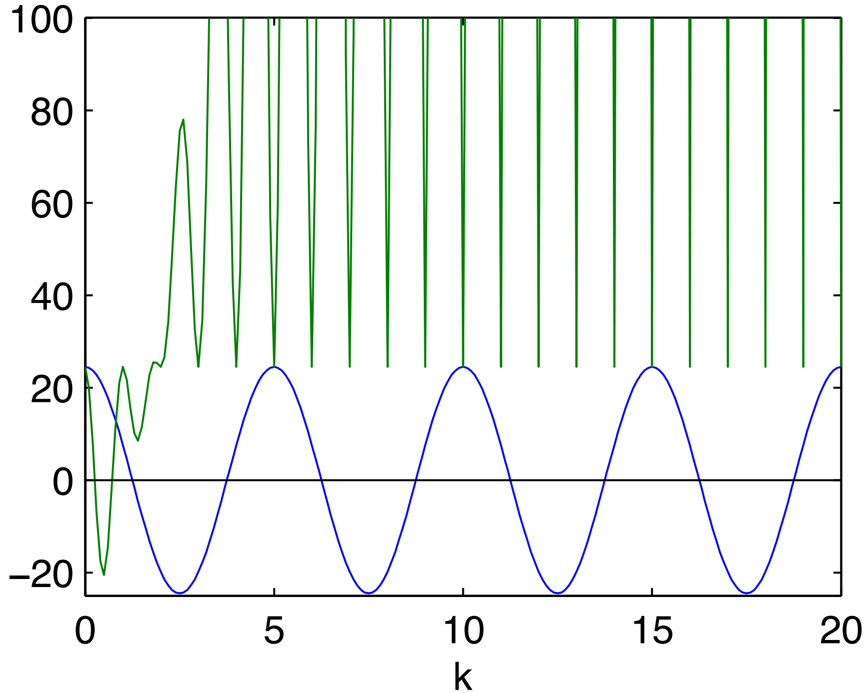
One-ring graph

Conditions:

$$\varphi_0(0) = \psi_0(\pi + \vartheta) \qquad \varphi_0(\pi - \vartheta) = \psi_0(0)$$
$$\varphi_0'(0) - \psi_0'(\pi + \vartheta) = -i\alpha\varphi_0(0) \qquad \psi_0'(0) - \varphi_0'(\pi - \vartheta) = i\alpha\psi_0(0).$$

Nontrivial solutions if $16k^2 \sin(k\pi)^2 - 4\alpha^2 \sin(k(\pi - \vartheta)) \sin(k(\pi + \vartheta)) = 0.$

The spectrum has two types of eigensolutions:



There is always a root below $k = |\alpha|/2$ and possibly states having zeros at positions of the point interaction.

Three-ring graph

Conditions: We have a system of 12 homogeneous equations for 12 unknowns $A_j^+, A_j^-, B_j^+, B_j^-$, $j = -1, 0, 1$. Requiring $\det(M_3) = 0$ we get eigenvalues. They are again of two types.

If $k = n \in \mathbb{N}$, then rank of M_3 is 10.

$$(A_{-1}^+ \sin(kx), A_{-1}^+ \sin(kx), 0, 0, A_1^+ \sin(kx), A_1^+ \sin(kx))$$