Mixed-state evolution in the presence of gain and loss

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Work based on:

- D. C. Brody & E. M. Graefe (2012)
  "Mixed-state evolution in the presence of gain and loss"
  (arXiv:1208.xxxx; to appear tomorrow).

Motivation

There have been a lot of interests in experimental realisations of PT-symmetric dynamics.

For a quantum system, it is difficult to balance gain and loss while maintaining quantum coherence.

We therefore wish to develop models that describe the evolution of mixed states in the presence of gain and loss.

Key questions

- What is the exact evolution equation for a mixed state when gain and loss are present?

- How does the presence of noise affects the evolution?

A naive approach

One might consider taking the von Neumann equation

$$\frac{\partial \rho}{\partial t} = -\mathrm{i}[H, \rho], \tag{1}$$

and make the replacement

$$H \to H - \mathrm{i}\Gamma \tag{2}$$

to obtain

$$\frac{\partial \rho}{\partial t} = -\mathrm{i}[H, \rho] - [\Gamma, \rho]. \tag{3}$$

However, the reality of ρ is not preserved...

Pure vs mixed states

- A dynamical equation for a pure state $= 2n - 2$ differential equations.

- A dynamical equation for a mixed state $= n^2 - 1$ differential equations.

$\Longrightarrow$ One cannot read off the 'correct' dynamical equation from the consideration of

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|. \tag{4}$$

Model for a mixed-state evolution

Our model is given by the dynamical equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}[H,\rho] - \big([\mathit{\Gamma},\rho]_+ - 2\,\mathrm{tr}(\rho\mathit{\Gamma})\rho\big), \tag{5}$$

where

$$H = H^\dagger$$

is the Hermitian part of the Hamiltonian generating ambient unitary motion,

$$\mathit{\Gamma} = \mathit{\Gamma}^\dagger$$

is the skew-Hermitian part of the Hamiltonian governing gain and loss, and

$$[\mathit{\Gamma},\rho]_+ = \mathit{\Gamma}\rho + \rho\mathit{\Gamma} \tag{6}$$

denotes the symmetric product.

We refer to (5) as the *covariance equation*.

Model for a mixed-state evolution (cont.)

The covariance equation is singled out on account of the fact that its formal solution can be expressed in the form

$$\rho_t = \frac{\mathrm{e}^{-\mathrm{i}(H-\mathrm{i}\Gamma)t}\rho_0 \mathrm{e}^{\mathrm{i}(H+\mathrm{i}\Gamma)t}}{\mathrm{tr}\left(\mathrm{e}^{-\mathrm{i}(H-\mathrm{i}\Gamma)t}\rho_0 \mathrm{e}^{\mathrm{i}(H+\mathrm{i}\Gamma)t}\right)}. \tag{7}$$

The dynamical equation satisfied by an observable $\langle F\rangle = \mathrm{tr}(F\rho_t)$ reads

$$\frac{\mathrm{d}\langle F\rangle}{\mathrm{d}t} = \mathrm{i}\langle [H,F]\rangle - \langle [\Gamma,F]_+\rangle + 2\,\langle\Gamma\rangle\langle F\rangle \tag{8}$$

This agrees with the complex extension of the Heisenberg equation of motion obtained in Graefe, *et al*. Phys. Rev. Lett. (2008) for pure states.

The covariance-type structure in (5) has also appeared in the contexts of:

(i) approach to thermal equilibrium: Korsch & Steffen, J. Phys. A**20**, (1987);

(ii) dissipative motion: Sergi & Zloshchastiev, arXiv:1207.4877; and

(ii) constrained quantum dynamics: Brody, *et al.* J. Phys. A**43**, (2010).

General properties of the model

(i) Overall probability conservation:

$$\partial_t \, \mathrm{tr}(\rho_t) = 0. \tag{9}$$

(ii) It does not in general preserve the purity of the state:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{tr}\, \rho^2 = -4\big(\mathrm{tr}(\Gamma\rho^2) - \mathrm{tr}(\rho\Gamma)\mathrm{tr}(\rho^2)\big). \tag{10}$$

(iii) The evolution equation (5) is *autonomous*.

(iv) The evolution speed is not a constant of motion and is given by the expression

$$\begin{aligned} v \;=\; & 2\left(\mathrm{tr}(H^2\rho) - \mathrm{tr}(H\sqrt{\rho}H\sqrt{\rho})\right) - 2\mathrm{i}\,\mathrm{tr}([H,\Gamma]\rho) \\ & +2\left(\mathrm{tr}(\Gamma^2\rho) + \mathrm{tr}(\Gamma\sqrt{\rho}\Gamma\sqrt{\rho}) - 2\big(\mathrm{tr}(\Gamma\rho)\big)^2\right), \end{aligned} \tag{11}$$

which for a pure state reduces to

$$v = 2\Delta H^2 + 2\Delta\Gamma^2 - 2\mathrm{i}\langle[\Gamma, H]\rangle. \tag{12}$$

Stationary states

(v) Every eigenstate of the Hamiltonian $K = H - \mathrm{i}\Gamma$ is a fixed point of the motion (5), and they are the only stationary states that are pure.

(vi) The mixed stationary states of the dynamical equation (5) consist of convex combinations of the eigenstates of K associated with *real* eigenvalues:

$$\rho_0 = \sum_{k=1}^{m} p_k |\phi_k\rangle\langle\phi_k|, \tag{13}$$

where $\{p_k\}$ are nonnegative numbers adding to unity, $\{|\phi_k\rangle\}$ are eigenstates of K with real eigenvalues, and m is the number of real eigenvalues.

(vii) The totality of mixed stationary states lies on the subspace: $\mathrm{tr}(\Gamma\rho) = 0$.

(viii) If the initial state ρ_0 admits an eigenfunction expansion such that one or more terms are associated with complex eigenvalues, then

$$\lim_{t\to\infty} \rho_t = |\phi^*\rangle\langle\phi^*|, \tag{14}$$

where $|\phi^*\rangle$ is the member of the eigenfunctions $\{|\phi_j\rangle\}$ for which the associated imaginary part of the eigenvalue γ_j takes maximum value.

PT-symmetric phase

In the PT symmetric phase where all eigenvalues are real, we put

$$\rho_0 = \sum_{j,k} \rho_{jk} |\phi_j\rangle\langle\phi_k|. \tag{15}$$

Then the trace condition $\mathrm{tr}(\rho_0) = 1$ implies that

$$\sum_{j=1}^{m} \rho_{jj} + \sum_{j \neq k} \rho_{jk} \langle\phi_k|\phi_j\rangle = 1. \tag{16}$$

The solution to the evolution equation then takes the form:

$$\rho_t = \frac{\sum_{j,k} \rho_{jk} \mathrm{e}^{-\mathrm{i}\omega_{jk}t} |\phi_j\rangle\langle\phi_k|}{\sum_{j,k} \rho_{jk} \mathrm{e}^{-\mathrm{i}\omega_{jk}t} \langle\phi_k|\phi_j\rangle}, \tag{17}$$

where $\omega_{jk} = E_j - E_k$.

Two-level example

We illustrate integral curves of (5) for a two-level example system with the Hamiltonian $K = \sigma_x - \mathrm{i}\gamma\sigma_z$, where γ is a real parameter.

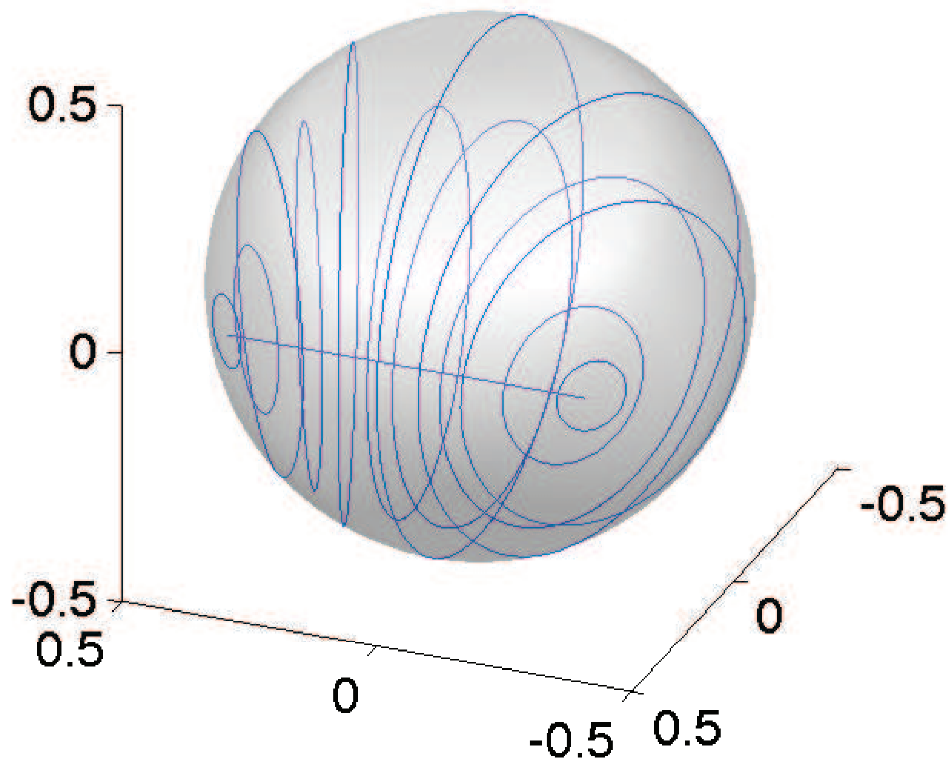

Figure 1: *Trajectories generated by the Hamiltonian* $K = \sigma_x - i\gamma\sigma_z$.

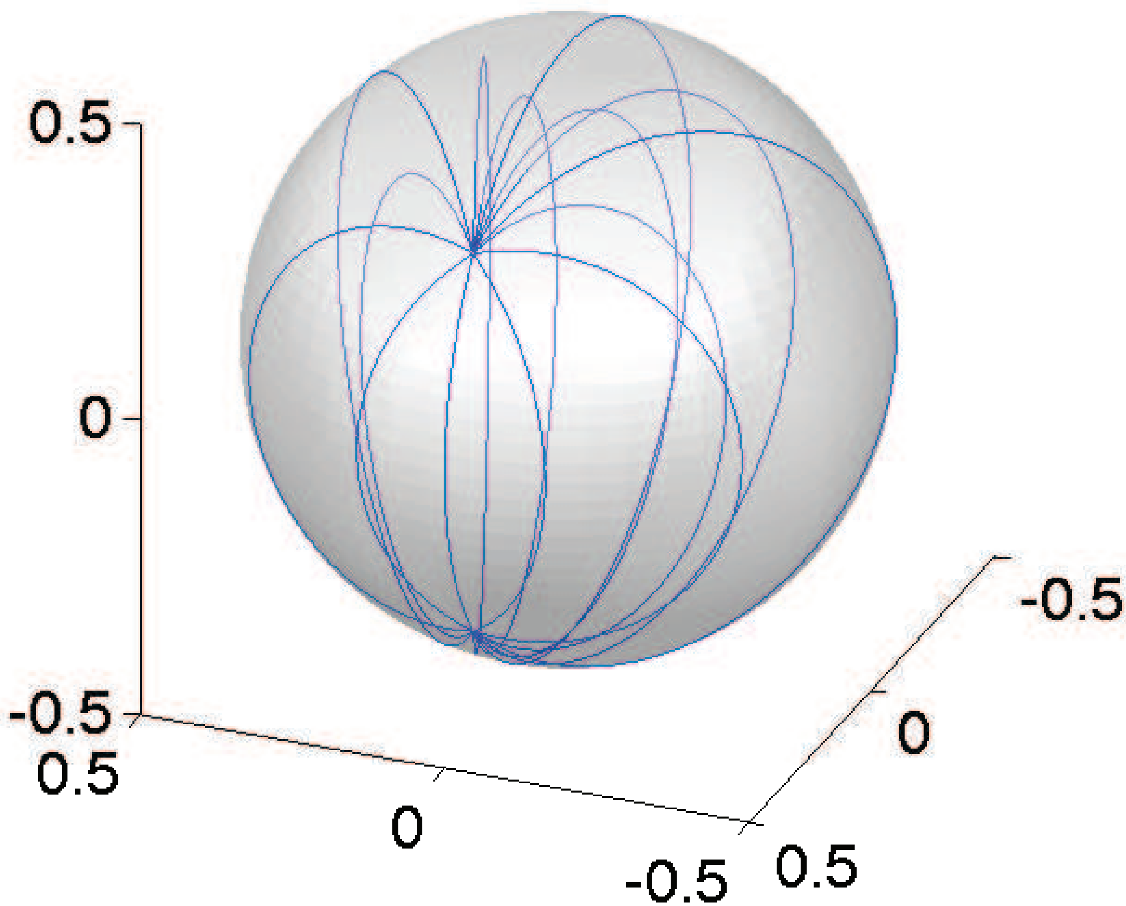

Figure 2: *Trajectories generated by the Hamiltonian* $K = \sigma_x - i\gamma\sigma_z$.

PT phase transition

Introduce an 'order parameter' m by the time average of σ_z:

$$m = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{tr}(\sigma_z \rho_t)\mathrm{d}t. \tag{18}$$

Then m is independent of the choice of the initial condition ρ_0.

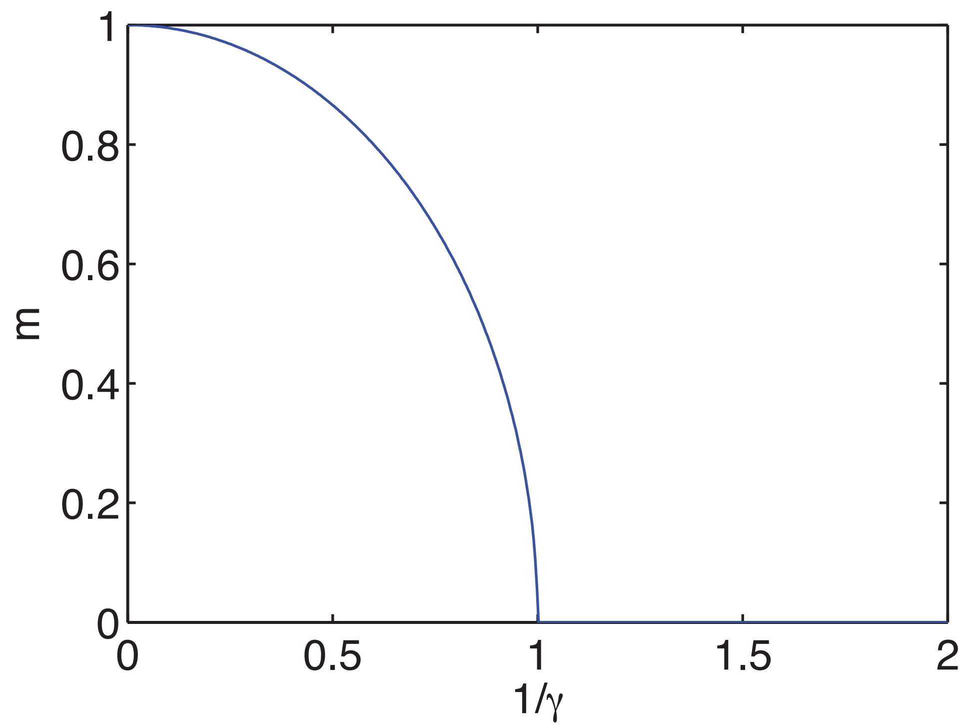

Figure 3: *Time averages of the observable σ_z as functions of γ^{-1}.*

Effect of noise

If a coherent implementation of gain and loss is not feasible, either because of fundamental quantum limits or current technological limits, then it is important to take into account additional effects arising from random perturbations.

This leads to the extended model:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}[H,\rho] - \big([\mathit{\Gamma},\rho]_+ - 2\,\mathrm{tr}(\rho\mathit{\Gamma})\rho\big) + \kappa\,(\mathbb{1} - n\rho)\,, \tag{19}$$

where $\kappa \geq 0$ and n is the Hilbert space dimensionality.

Properties of the noisy extension

(i) The evolution equation (19) preserves the overall probability.

(ii) The evolution of the purity is governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{tr}\,\rho^2 = -4\big(\mathrm{tr}(\mathit{\Gamma}\rho^2) - \mathrm{tr}(\rho\mathit{\Gamma})\mathrm{tr}(\rho^2)\big) + 2\kappa\big(1 - n\,\mathrm{tr}(\rho^2)\big). \tag{20}$$

Hence an initially pure state necessarily evolves into a mixed state.

(iii) When $\Gamma = 0$, the solution to the dynamical equation (19) takes the form

$$\rho_t = \frac{1}{n}\left[\mathbb{1} + \left(n\,\mathrm{e}^{-\mathrm{i}Ht}\rho_0\mathrm{e}^{\mathrm{i}Ht} - \mathbb{1}\right)\mathrm{e}^{-\kappa nt}\right], \qquad (21)$$

and has a single fixed point $\rho_\infty = n^{-1}\mathbb{1}$.

(iv) The evolution equation (19) is autonomous.

(v) The dynamical equation satisfied by an observable $\langle F\rangle = \mathrm{tr}(F\rho_t)$ is given by

$$\frac{\mathrm{d}\langle F\rangle}{\mathrm{d}t} = \mathrm{i}\langle[H,F]\rangle - \langle[\Gamma,F]_+\rangle + 2\,\langle\Gamma\rangle\langle F\rangle + \kappa\big(\mathrm{tr}(F) - n\langle F\rangle\big). \qquad (22)$$

Emergence of equilibrium states

Numerical studies indicate that all initial state ρ_0 evolve into an equilibrium state ρ_∞ that is similar to a canonical density matrix of quantum statistical mechanics.

The effect of equilibrium states is to suppress the PT phase transition.

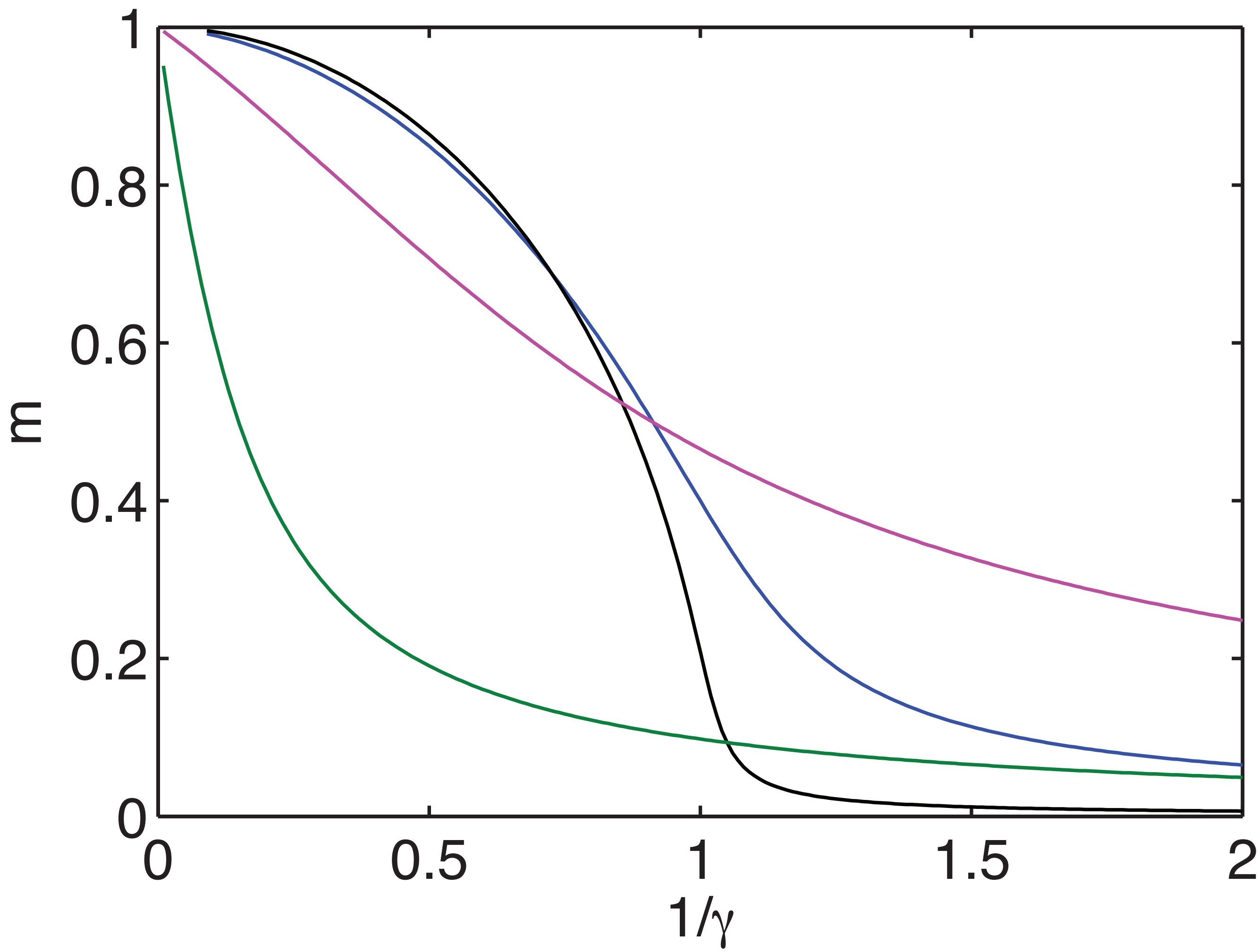

Figure 4: *Time and ensemble averages of the observable* σ_z *as functions of* γ^{-1}.