NONUNIQUENESS OF THE $\mathcal{C}$ OPERATOR IN $\mathcal{PT}$-SYMMETRIC QUANTUM MECHANICS

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-Work joint with Carl Bender-

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Conclusions

- We have constructed a nonunique $\mathcal{C}$ operator, and we have found that the nonuniqueness of $\mathcal{C}$ is associated with the unboundedness of the metric operator.
- In particular, for the simple case of the harmonic oscillator, we have constructed infinite unbounded $\mathcal{C}$ operators.
- Unfortunately, for other $\mathcal{PT}$-symmetric Hamiltonians the explicit evaluation of closed form expression for the coefficients in the series expansion of $\mathcal{C}$ is extremely complicated, even for the simple case of the shifted harmonic oscillator....

Unboundedness of the $\mathcal{C}$ operator is an hot topic in $\mathcal{PT}$-symmetric quantum mechanics.

$\mathcal{C}$ operator in $\mathcal{PT}$-symmetric quantum mechanics

A $\mathcal{PT}$-symmetric Hamiltonian having an unbroken $\mathcal{PT}$ symmetry defines a physical theory of quantum mechanics.

A linear operator $\mathcal{C}$ can be constructed, it represents an hidden symmetry of the $\mathcal{PT}$-symmetric Hamiltonian.

- ✓ In terms of $\mathcal{C}$, an inner product with a positive norm can be defined:

$$\boxed{\langle\psi|\zeta\rangle^{CPT} = \int dx\, \psi^{CPT}(x)\zeta(x)} \qquad \psi^{CPT}(x) = \int dy\, \mathcal{C}(x,y)\psi^*(y)$$

- ✓ The time evolution of the theory is unitary (norm is preserved in time).[1]

$\mathcal{C}$ obeys the following three algebraic equations:

$$\boxed{\mathcal{C}^2 = 1, \qquad [\mathcal{C}, \mathcal{PT}] = 0, \qquad [\mathcal{C}, H] = 0.}$$

[1] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002);
C. M. Bender, Rept. Prog. Phys. **70**, 947 (2007).

$\mathcal{C}$ operator in $\mathcal{PT}$-symmetric quantum mechanics

$$\boxed{\mathcal{C}^2 = 1, \qquad [\mathcal{C}, \mathcal{PT}] = 0, \qquad [\mathcal{C}, H] = 0.}$$

- The $\mathcal{C}$ operator for a few nontrivial quantum-mechanical models has been calculated exactly.
  - **Lee model**, *C. M. Bender, S. F. Brandt, J. H. Chen, and Q. Wang, Phys. Rev. D* **71***, 025014 (2005).*
  - $-x^4$ **Potenital**, *H. F. Jones and J. Mateo, Phys. Rev. D* **73***, 085002 (2006).*
  - **Pais-Uhlenbeck oscillator**, *C. M. Bender and P. D. Mannheim, Phys. Rev. Lett.* **100***, 110402 (2008).*

- In general this system of equations is extremely difficult to solve analytically: In most cases a perturbative approach has been adopted.
  - ix^3 **Potential,** *C. M. Bender, P. N. Meisinger, and Q. Wang, J. Phys. A: Math. Gen.* **36***, 1973 (2003); C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. D* **70***, 025001 (2004);*
  - ix^2y **and** $i\,xyz$ **Potentials,** *C. M. Bender, J. Brod, A. Refig, and M. E. Reuter, J. Phys. A: Math. Gen.* **37***, 10139-10165 (2004);*
  - $\mathcal{PT}$**-symmetric square well,** *C. M. Bender and B. Tan, J. Phys. A: Math. Gen.* **39***, 1945 (2006).*

Calculation of the $\mathcal{C}$ operator: Standard Approach

- Express the $\mathcal{C}$ operator as an exponential of a Dirac Hermitian operator Q multiplied by the parity operator $\mathcal{P}$:

$$\boxed{\mathcal{C} = e^{Q}\mathcal{P}}$$

$e^{Q/2}$ is the metric operator η that can be used to transform the non-Hermitian Hamiltonian H to an isospectral Hermitian Hamiltonian.

A. Mostafazadeh, J. Math. Phys. **43***, 205 (2002), J. Phys. A: Math. Gen.* **36***, 7081 (2003);*
F. Scholtz, H. Geyer, and F. Hahne, Ann. Phys. **213***, 74 (1992).*

- Solution to the first two equations:

$$\boxed{\mathcal{C}^2 = 1, \qquad [\mathcal{C}, \mathcal{PT}] = 0} \longrightarrow \boxed{Q(x,p) = Q(-x,p), \quad Q(x,p) = -Q(x,-p)}$$

- Solution to the third equation:

$$\boxed{[\mathcal{C}, H] = 0}$$

Calculation of the $\mathcal{C}$ operator: Standard Approach

PERTURBATIVE APPROACH - treat ϵ as a small parameter:

$$\boxed{H = H_0 + \epsilon H_1}$$

H_0 commutes with $\mathcal{P}$ and H_1 anticommutes with $\mathcal{P}$.

Quantum-mechanical cases that have been studied: $H = H_0 + i\epsilon q$ and $H = H_0 + i\epsilon q^3$, where H_0 is the harmonic-oscillator Hamiltonian.

$$\boxed{[\mathcal{C}, H] = 0} \longrightarrow \boxed{[e^Q, H_0] = \epsilon \left\{ e^Q, H_1 \right\}}$$

$$\boxed{Q = \epsilon Q_1 + \epsilon^3 Q_3 + \epsilon^5 Q_5 + \ldots}$$

Why odd powers of ϵ ?

$$[H_0, Q_0] = 0$$

$$\begin{aligned}
[H_0, Q_1] &= -2H_1 \\
[H_0, Q_2] &= -\frac{1}{2}\left[H_0, Q_1^2\right] + \{Q_1, H_1\} \\
[H_0, Q_3] &= -\left[H_0, \frac{1}{6}Q_1^3 + \frac{1}{2}\{Q_1, Q_2\}\right] + \left\{\tfrac{1}{2}Q_1^2 + Q_2, H_1\right\} \\
[H_0, Q_4] &= -\left[H_0, \frac{1}{24}Q_1^4 + Q_1\{Q_1, Q_2\} + Q_2Q_1^2 + \frac{1}{2}\{Q_1, Q_3\}\right] + \left\{\frac{1}{6}Q_1^3 + \frac{1}{2}\{Q_1, Q_2\} + Q_3, H_1\right\}
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Why odd powers of ϵ ?

$[H_0, Q_0] = 0$ *Q_0 must be a function of H_0...* $\boxed{Q_0 = 0}$

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\end{aligned}$$

$$\boxed{Q_2 = 0}$$

Calculation of the $\mathcal{C}$ operator: Standard Approach

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$$\boxed{[\mathcal{C}, H] = 0} \qquad \boxed{[e^Q, H_0] = \epsilon \left\{ e^Q, H_1 \right\}}$$

$$\boxed{Q = \epsilon Q_1 + \epsilon^3 Q_3 + \epsilon^5 Q_5 + \ldots}$$

Why odd powers of ϵ ?

$[H_0, Q_0] = 0$ *Q_0 must be a function of H_0...* $\boxed{Q_0 = 0}$

$$\begin{aligned}
[H_0, Q_1] &= -2H_1 \\
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\end{aligned}$$

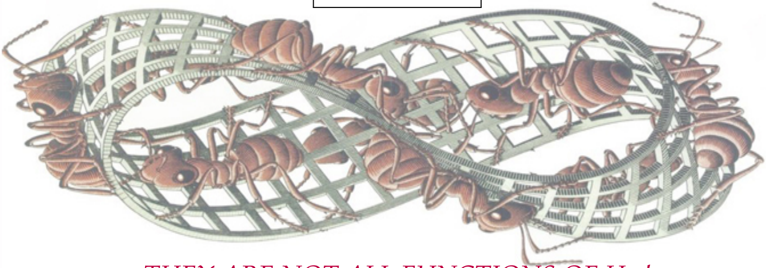
$\boxed{Q_2 = 0}$

$\boxed{Q_4 = 0}$

Nonuniqueness of the $\mathcal{C}$ operator

There are an infinite number of one-parameter families of solutions to the equation :

$$\boxed{[H_0, Q_k] = 0}$$



THEY ARE NOT ALL FUNCTIONS OF H_0 !

Many of them are odd in p and even in q.

It is precisely because of the existence of these solutions that **the $\mathcal{C}$ operator is nonunique**.

Nonuniqueness of the $\mathcal{C}$ operator

- In *C. M. Bender and S. P. Klevansky, Phys. Lett. A* **373**, *2670 (2009)* the existence of multiple solutions to the commutator equation

$$[\mathcal{C}, H] = 0$$

has been discussed for the first time:

$$H = H_0 + \epsilon H_1, \qquad Q = \sum_{k=0}^{\infty} \epsilon^{2k+1} Q_{2k+1}$$

$$[Q_1, H_0] = -2H_1$$

$$\boxed{Q_1 = Q_1^{\min} + Q_1^{\text{hom}}}$$

Minimal solution: consisting in the simplest particular solution satisfying the recursion relation generated from the expansion coefficients $\alpha_{m,n}$ in the series $Q = \sum_{m,n} \alpha_{m,n} T_{m,n}$.

Solutions to the homogeneous equation

$$[H_0, Q_1] = 0.$$

It consist in infinite-parameters classes of mathematically acceptable solutions.

A new strategy to calculate $\mathcal{C}$ operators

Within this approach, it is extremely hard to calculate a closed form expression for the next order solutions $Q_3, Q_5, Q_7, \ldots$

- *Finding multiple solutions for Q_1 is not the only way to demonstrate nonuniqueness of $\mathcal{C}$!*
- *There is a clearer and more fundamental way to explain the nonuniqueness of the $\mathcal{C}$ operator...*

$$\boxed{Q_0 \neq 0}$$

A new strategy to calculate $\mathcal{C}$ operators

Consider a perturbative approach in the small parameter ϵ:

$$\boxed{H = H_0 + \epsilon H_1}$$

H_0 commutes with $\mathcal{P}$ and H_1 anticommutes with $\mathcal{P}$.

$$\boxed{[\mathcal{C}, H] = 0} \qquad \boxed{[e^Q, H_0] = \epsilon \left\{ e^Q, H_1 \right\}}$$

$$\boxed{Q = \epsilon Q_1 + \epsilon^3 Q_3 + \epsilon^5 Q_5 + \ldots}$$

Why odd powers of ϵ ?

It is only one possible solution!

$[H_0, Q_0] = 0$ *Q_0 must be a function of H_0...* $\boxed{Q_0 = 0}$

WRONG !

$$\begin{aligned}
[H_0, Q_1] &= -2H_1 \\
[H_0, Q_2] &= -\frac{1}{2}\left[H_0, Q_1^2\right] + \{Q_1, H_1\} \\
[H_0, Q_3] &= -\left[H_0, \frac{1}{6}Q_1^3 + \frac{1}{2}\{Q_1, Q_2\}\right] + \left\{\tfrac{1}{2}Q_1^2 + Q_2, H_1\right\} \\
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\end{aligned}$$

A new strategy to calculate $\mathcal{C}$ operators

WHERE IS THE ADVANTAGE IN THE CHOICE $Q_0 \neq 0$?

- A more general way to represent Q includes all the power of ϵ:

$$Q(p,q) = \sum_{j=0}^{\infty} \epsilon^j Q_j(p,q).$$

IN THE LIMIT $\epsilon \to 0$ WE OBTAIN AN INFINITE CLASS OF EXACT $\mathcal{C}$ OPERATORS FOR THE QUANTUM HARMONIC OSCILLATOR HAMILTONIAN H_0

$$[\mathcal{C}, H] = 0, \qquad H = H_0 + \cancel{\epsilon H_1}$$

$$\mathcal{C} = e^{Q_0 \cancel{+\epsilon Q_1 + \epsilon^2 Q_2 + \dots}}\mathcal{P} \qquad [Q_0, H_0] = 0.$$

Then, once that we have Q_0 for the harmonic oscillator case, we can straightforwardly generalize this result and claim that the operator $\mathcal{C}$ is nonunique.

Working tools

$$\boxed{\mathcal{Q} = \sum_{m,n} \alpha_{m,n} T_{m,n}}$$

The basis elements $T_{m,n}$ are the quantum-mechanical generalization of the classical product $p^m q^n$. They are defined as a totally symmetric averages of all possible orderings of m factors of p and n factors of q.

$$\begin{aligned}
T_{0,0} &= 1, \\
T_{1,0} &= p \\
T_{1,1} &= \frac{1}{2}(pq + qp), \\
T_{1,2} &= \tfrac{1}{3}(pqq + qpq + qqp), \\
T_{2,2} &= \tfrac{1}{6}(ppqq + qqpp + pqqp + qppq + qpqp + pqpq).
\end{aligned}$$

They can be recast in Weyl-ordered form:

$$\boxed{T_{m,n} = \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} p^k\, q^n p^{m-k} = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} q^k\, p^m q^{n-k}.}$$

Working tools

- The $T_{m,n}$ operators form an algebra closed under multiplication
- The $T_{m,n}$ operators obey simple commutation and anticommutation relations, that continue to hold in the extended singular basis with negative values of m and n:

$$[p, T_{m,n}] = -i\,n\,T_{m,n-1}, \qquad [q, T_{m,n}] = i\,m\,T_{m-1,n},$$

$$\{p, T_{m,n}\} = 2T_{m+1,n}, \qquad \{q, T_{m,n}\} = 2T_{m,n+1},$$

$$\left[p^2, T_{m,n}\right] = -2inT_{m+1,n-1}, \quad \left[q^2, T_{m,n}\right] = 2imT_{m-1,n+1}.$$

- Useful identities[2] for $m = \pm n$:

$$\boxed{T_{n,n} = \frac{1}{(2n-1)!!} S_n(T_{1,1})} \qquad \boxed{T_{-n,n} = \frac{1}{2}\left(\frac{1}{p}q\right)^n + \frac{1}{2}\left(q\frac{1}{p}\right)^n}$$

[2] C. M. Bender, L. R. Mead, and S. S. Pinsky, J. Math. Phys. **28**, 509 (1987); C. M. Bender and S. P. Klevansky, Phys. Lett. A **373**, 2670 (2009).

Solutions to the homogeneous equation

$$\boxed{Q_0 = \sum_{m,n} a_{m,n} T_{m,n}} \quad \Longrightarrow \quad \boxed{[Q_0, H_0] = 0}$$

Solutions odd in p and even in q have the general form [3]:

$$Q_0^{(\gamma)}(p,q) = \sum_{k=0}^{\infty} a_k^{(\gamma)} T_{2\gamma+1-2k,2k}, \quad \gamma = 0, \pm 1, \pm 2, \ldots$$

where the coefficients $a_k^{(\gamma)}$ satisfy the following two-term recursion relation:

$$a_{k+1}^{(\gamma)}(k+1) - (\gamma - k + 1/2)a_k^{(\gamma)} = 0,$$

$$a_k^{(\gamma)} = a_0^{(\gamma)}(-1)^k \frac{\Gamma(k-\gamma-1/2)}{k!\Gamma(-\gamma-1/2)} \quad (k = 0, 1, 2, \ldots)$$

The series can be summed as a binomial expansion:

$$Q_0^{(\gamma)} = \frac{a_0^{(\gamma)}}{2^{2\gamma+2}} \left\{ \ldots \left\{ \left\{ \left(1 + q\frac{1}{p}q\frac{1}{p}\right)^{\gamma+1/2} + \left(1 + \frac{1}{p}q\frac{1}{p}q\right)^{\gamma+1/2}, p \right\}, p \right\} \ldots, p \right\}_{(2\gamma+1)\ \text{times}}$$

[3] *C. M. Bender and S. P. Klevansky, Phys. Lett. A* **373**, *2670 (2009); C. M. Bender and MG, J. Math. Phys.* **53**, *062102 (2012)*

$\mathcal{C}$ operators for the Harmonic oscillator Hamiltonian

$$\mathcal{C}^2 = 1, \qquad [\mathcal{C}, \mathcal{PT}] = 0, \qquad [\mathcal{C}, H_0] = 0.$$

$$H_0 = \tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 \qquad \boxed{\mathcal{C} = e^{Q_0}\mathcal{P}}$$

$$\boxed{Q_0^{(\gamma)} = \frac{a_0^{(\gamma)}}{2^{2\gamma+2}} \left\{ \dots \left\{ \left\{ \left(1 + q\frac{1}{p}q\frac{1}{p}\right)^{\gamma+1/2} + \left(1 + \frac{1}{p}q\frac{1}{p}q\right)^{\gamma+1/2}, p \right\}, p \right\} \dots, p \right\}_{(2\gamma+1)\ \text{times}}}$$

- While the construction of the solutions involves series in inverse powers of p, these solutions are well behaved as $p \to 0$.

Solution corresponding to $\underline{\gamma = 0}$:

$$\boxed{Q_0^{(0)} = \frac{1}{4}a_0^{(0)} \left(\sqrt{1 + q\frac{1}{p}q\frac{1}{p}}\, p + p\sqrt{1 + q\frac{1}{p}q\frac{1}{p}} + \sqrt{1 + \frac{1}{p}q\frac{1}{p}q}\, p + p\sqrt{1 + \frac{1}{p}q\frac{1}{p}q} \right).}$$

- In the classical limit for which p and q become commuting numbers:

$$Q_{0,\text{classical}}^{(0)} = a_0^{(0)}\,\mathrm{sgn}(p)\sqrt{p^2 + q^2}$$

Another example: solution for $\underline{\gamma = -1}$:

$$\boxed{Q_0^{(-1)} = \frac{a_0^{(-1)}}{2i} \left[\hat{p}, \operatorname{arcsinh}\left(\hat{q}\frac{1}{\hat{p}}\right) + \operatorname{arcsinh}\left(\frac{1}{\hat{p}}\hat{q}\right) \right]}$$

It has recently become clear that the nonuniqueness of the $\mathcal{C}$ operator has important implications for the mathematical and physical interpretation of $\mathcal{PT}$-symmetric quantum mechanics.

$$\boxed{Q_0^{(\gamma)} = \frac{a_0^{(\gamma)}}{2^{2\gamma+2}} \underbrace{\left\{\ldots\left\{\left\{\left(1+q\frac{1}{p}q\frac{1}{p}\right)^{\gamma+1/2} + \left(1+\frac{1}{p}q\frac{1}{p}q\right)^{\gamma+1/2}, p\right\}, p\right\}\ldots, p\right\}}_{(2\gamma+1)\text{ times}}}$$

- For the harmonic oscillator, the metric operator $\eta = e^{Q_0}$ is just the unity when $Q_0 = 0$. In this special case, the metric operator is *bounded*.
- However, since there is an infinite number of possible choices for Q_0, there is an infinite number of possible metric operators. The metric operators $\eta = e^{Q_0}$ for $Q_0 \neq 0$ are *unbounded*.

Recent papers on the unboundedness of the $\mathcal{C}$ operator:

- C. M. Bender and S. Kuzhel, J. Phys. A: Math. Theor. (in press).
- F. Bagarello and M. Znojil, J. Phys. A: Math. Theor. **45**, 115311 (2012);
- A. Mostafazadeh, ArXiv:1203.6241v4;
- B. Samsonov, (to be published in Special J. Phys. A Issue).

Calculation of Q to the first order in ϵ

Let us concentrate on the solution to the formal problem of determining Q to the first-order in ϵ once that Q_0 is given:

$$\mathcal{C} = e^{Q_0+\epsilon Q_1+\epsilon^2 Q_2+\epsilon^3 Q_3+\dots}\,\mathcal{P}$$

The coefficient Q_1 satisfies the equation:

$$\boxed{\epsilon\left\{e^{Q_0}, H_1\right\} = \left[e^{Q_0+\epsilon Q_1}, H_0\right]}$$

$$\begin{aligned}\left\{e^{Q_0}, H_1\right\} &= Z + \frac{1}{2}(Q_0 Z + Z Q_0) + \frac{1}{6}(Q_0^2 Z + Q_0 Z Q_0 + Z Q_0^2) \\ &+ \frac{1}{24}(Q_0^3 Z + Q_0^2 Z Q_0 + Q_0 Z Q_0^2 + Z Q_0^3) + \dots\end{aligned}$$

where we have defined the operator:

$$Z \equiv [Q_1, H_0]\,.$$

Calculation of Q to the first order in ϵ

" Try to solve a problem exactly,
if not try to find a small parameter,
if not try again."

L. D. Landau

Calculation of Q to the first order in ϵ

- Q_0 is a solution to the *homogeneous* equation $[Q_0, H_0] = 0$. We can use the scale invariance of Q_0:

$$Q_0 \rightarrow \mu Q_0$$

Treat μ as a small perturbation parameter into the first order in ϵ equation for Q_1:

$$\begin{aligned} \left\{e^{Q_0}, H_1\right\} &= Z + \frac{1}{2}(Q_0 Z + Z Q_0) + \frac{1}{6}(Q_0^2 Z + Q_0 Z Q_0 + Z Q_0^2) \\ &+ \frac{1}{24}(Q_0^3 Z + Q_0^2 Z Q_0 + Q_0 Z Q_0^2 + Z Q_0^3) + \ldots \end{aligned}$$

$$Z = \sum_{n=0}^{\infty} Z_n \mu^n, \qquad \boxed{Z = [Q_1, H_0]}$$

The general result can be given in terms of Bernoulli numbers $\mathcal{B}_n$:

$$\boxed{Z_n = \frac{2\mathcal{B}_n}{n!}\,[Q_0, \ldots [Q_0, [Q_0, H_1]] \ldots]_{n \text{ times}}\,, \quad n \neq 1}$$

$$Q_1 = \sum_{n=0}^{\infty} Q_{1,n} \mu^n,$$

Each order in Q_1 is solution of the commutator equation

$$Z_n = \left[Q_{1,n}, H_0\right], \quad n = 0, 1, 2, \ldots$$

Q_1 for the shifted harmonic oscillator

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 + \epsilon\, iq$$

- H has an unbroken $\mathcal{PT}$ symmetry for all real ϵ.
- Its real eigenvalues are:

$$E_n = n + \frac{1}{2} + \frac{1}{2}\epsilon^2, \qquad (n = 0,\, 1,\, 2,\, \ldots)$$

- One $\mathcal{C}$ operator for this theory is given exactly by:

$$\mathcal{C} = e^{-2\epsilon p}\mathcal{P}$$

C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89***, 270401 (2002); C. M. Bender, Rept. Prog. Phys.* **70***, 947-1018 (2007).*

However, the solution for $\mathcal{C}$ is not unique!

Recipe for the construction of Q_1 for the shifted harmonic oscillator

INGREDIENTS:

- Take one or more solutions Q_0.

  For simplicity, we choose

$$Q_0 = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k-1/2)}{\Gamma(k)} T_{1-2k,2k}$$

- Remember the algebra of the $T_{m,n}$ operators

$$[T_{m,n}, T_{r,s}] = 2\sum_{j=0}^{\infty}\sum_{\ell=0}^{j}(-1)^\ell \frac{(i/2)^{2j+1}\Gamma(m+1)\Gamma(n+1)\Gamma(r+1)\Gamma(s+1)}{(2j+1)!\Gamma(m-\ell+1)\Gamma(n+\ell-2j)\Gamma(r+\ell-2j)\Gamma(s-\ell+1)} T_{m+n-2j-1,r+s-2j-1}$$

- Have some patience, and start to evaluate:

$$\begin{aligned}
[Q_{1,0}, H_0] &= 2i\,q, \\
[Q_{1,2}, H_0] &= \frac{i}{6}\,[Q_0, [Q_0, q]]\,, \\
[Q_{1,4}, H_0] &= -\frac{i}{360}\,[Q_0, [Q_0, [Q_0, [Q_0, q]]]] \\
\ldots &\quad \ldots
\end{aligned}$$

Q_1 for the shifted harmonic oscillator

$\boxed{\mu^0}$ For $n = 0$ we have a simple exact solution to the commutator equation for $Q_{0,1}$:

$$[Q_{1,0}, H_0] = 0 \qquad \Longrightarrow \qquad Q_{1,0} = -2p.$$

$\boxed{\mu^2}$ The equation for $Q_{1,2}$ is

$$[Q_{1,2},\, H_0] = \frac{i}{6}\sum_{k=1}^{\infty}\sum_{\alpha=0}^{\infty}(-1)^{k-\alpha+1}\frac{\Gamma(k+2\alpha)\,\Gamma(\alpha+1/2)^2}{\Gamma(k)\,\Gamma(\alpha+1)^2}T_{-2k-4\alpha,2k-1}.$$

This is a linear equation, we can solve it for each α separately and express the solution as a sum over α...
One solution is:

$$\boxed{Q_{1,2} = \sum_{\alpha=0}^{\infty}\sum_{k=1}^{\infty}\rho_{k,\alpha}T_{-2k-4\alpha-1,2k}}$$

where the coefficients $\rho_{k,\alpha}$ satisfy the recursion relation:

$$(2k+4\alpha-1)\rho_{k-1,\alpha} + 2k\rho_{k,\alpha} = A_{k,\alpha}$$

whose solution is

$$\boxed{\rho_{k,\alpha} = (-1)^k\frac{(k+2\alpha)!}{k!}\left(\frac{\Gamma(\alpha+1/2)}{\alpha!}\right)^2}$$

Technical details on the evaluation of $Q_{1,2}$

- By using the algebra of $T_{m,n}$ operators, the equation that $Q_{1,2}$ must solve is :

$$\left[Q_{1,2},\, H_0\right] = \frac{i}{6}\sum_{\alpha=0}^{\infty}\sum_{k=1}^{\infty}(-1)^{k-\alpha+1}\frac{\Gamma\left(k+2\alpha\right)\Gamma\left(\alpha+1/2\right)^2}{\Gamma\left(k\right)\Gamma\left(\alpha+1\right)^2}T_{-2k-4\alpha,2k-1}. \tag{1}$$

- It is a linear equation, so we solve it for each α separately and express the solution as a sum over α: $Q_{1,2} = \sum_{\alpha=0}^{\infty} Q_{1,2}^{\alpha}$.
- For general α we expand $Q_{1,2}^{\alpha}$ into the basis of the $T_{m,n}$ operators :

$$Q_{1,2}^{\alpha} = \sum_{m,n}\rho_{m,n}T_{m,n} \tag{2}$$

- The commutator between (2) and H_0 gives:

$$\left[Q_{1,2},\, H_0\right] = \sum_{m,n}\left[(m+1)\rho_{m+1,n-1} + (n+1)\rho_{m-1,n+1}\right]T_{m,n} \tag{3}$$

- Now we choose the *minimal* solution: let be $m = -n - 4\alpha - 1$ for $n \geq 1$, and make the substitution $n = 2k - 1$ into (3).
- Comparing (3) with (1) we obtain the recursion relation for the coefficients $\rho_{k,\alpha}$:

$$(2k + 4\alpha - 1)\rho_{k-1,\alpha} + 2k\rho_{k,\alpha} = A_{k,\alpha}. \tag{4}$$

One solution of (4) is:

$$\boxed{\rho_{k,\alpha} = (-1)^k\frac{(k+2\alpha)!}{k!}\left(\frac{\Gamma\left(\alpha+1/2\right)}{\alpha!}\right)^2} \tag{5}$$

Q_1 for the shifted harmonic oscillator

The operator :

$$Z_n = \frac{2\mathcal{B}_n}{n!}\,[Q_0,\ldots[Q_0,[Q_0,H_1]]\ldots]_{n\ \text{times}}\,, \qquad n \neq 1$$

for $H_1 = iq$ can be written in the general form:

$$Z_{2n} = \sum_{k=n}^{\infty}\sum_{\alpha=0}^{\infty} A_\alpha^{(2n)}(-1)^k \frac{\Gamma(k+2\alpha+n-1)}{\Gamma(k)} T_{-2k-4\alpha,2k-2n+1}$$

The explicit form of the operators $Q_{1,2n}$ that satisfies the equation $\left[Q_{1,2n}, H_0\right] = Z_{2n}$ is:

$$\boxed{Q_{1,2n} = \sum_{\alpha=0}^{\infty}\sum_{k=n-1}^{\infty} \rho_{k,\alpha}^{(2n)} T_{-2k-4\alpha-1,2k-2n+2}}$$

The recursion satisfied by the coefficients $\rho_{k,\alpha}$ is:

$$(2k+4\alpha-1)\rho_{k-1,\alpha}^{(2n)} + 2(k-n+1)\rho_{k,\alpha}^{(2n)} = (-1)^k \frac{\Gamma(k+2\alpha+n-1)}{\Gamma(k)} A_\alpha^{(2n)}$$

whose solution is:

$$\boxed{\rho_{k,\alpha}^{2n} = \frac{(-1)^k\Gamma(k+2\alpha+3/2)}{\Gamma(k+3-n)\Gamma(2\alpha+n+1/2)}\left(G_\alpha + A_\alpha^{2n} - \frac{A_\alpha^{2n}\Gamma(2\alpha+3/2)\Gamma(2\alpha+k+2)}{\Gamma(2\alpha+k+3/2)}\right).}$$

Conclusions

- We have constructed a nonunique $\mathcal{C}$ operator, and we have found that the nonuniqueness of $\mathcal{C}$ is associated with the unboundedness of the metric operator.
- In particular, for the simple case of the harmonic oscillator, we have constructed infinite unbounded $\mathcal{C}$ operators.
- Unfortunately, for other $\mathcal{PT}$-symmetric Hamiltonians the explicit evaluation of closed form expression for the coefficients in the series expansion of $\mathcal{C}$ is extremely complicated, even for the simple case of the shifted harmonic oscillator....

Unboundedness of the $\mathcal{C}$ operator is an hot topic in $\mathcal{PT}$-symmetric quantum mechanics.

Conclusions

The importance of being unbounded

- The fact that the $\mathcal{C}$ operator is unbounded is signicant because, while there is a formal mapping between the Hilbert spaces of the two theories, the mapping does not map all of the vectors in the domain of one Hamiltonian into the domain of the other Hamiltonian.
- Consequently, even if the conventionally Hermitian Hamiltonian and the $\mathcal{PT}$ -symmetric Hamiltonian are isospectral, they are two mathematically distinct theories. [4]

At a fundamental mathematical level a $\mathcal{PT}$ -symmetric Hamiltonian describes a theory that is new.

[4] *C. M. Bender and S. Kuzhel, J. Phys. A: Math. Theor. (in press).*

Thanks for your attention!