Universal Spectral Behavior of $x^2(ix)^\varepsilon$ Potentials
(work with Carl Bender)

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In 1998, Bender and Boettcher studied a class of Hamiltonians:

$$H = p^2 + x^2(ix)^\varepsilon$$

where ε is taken as a real parameter.

The prescription to examine this Hamiltonian is well-known:

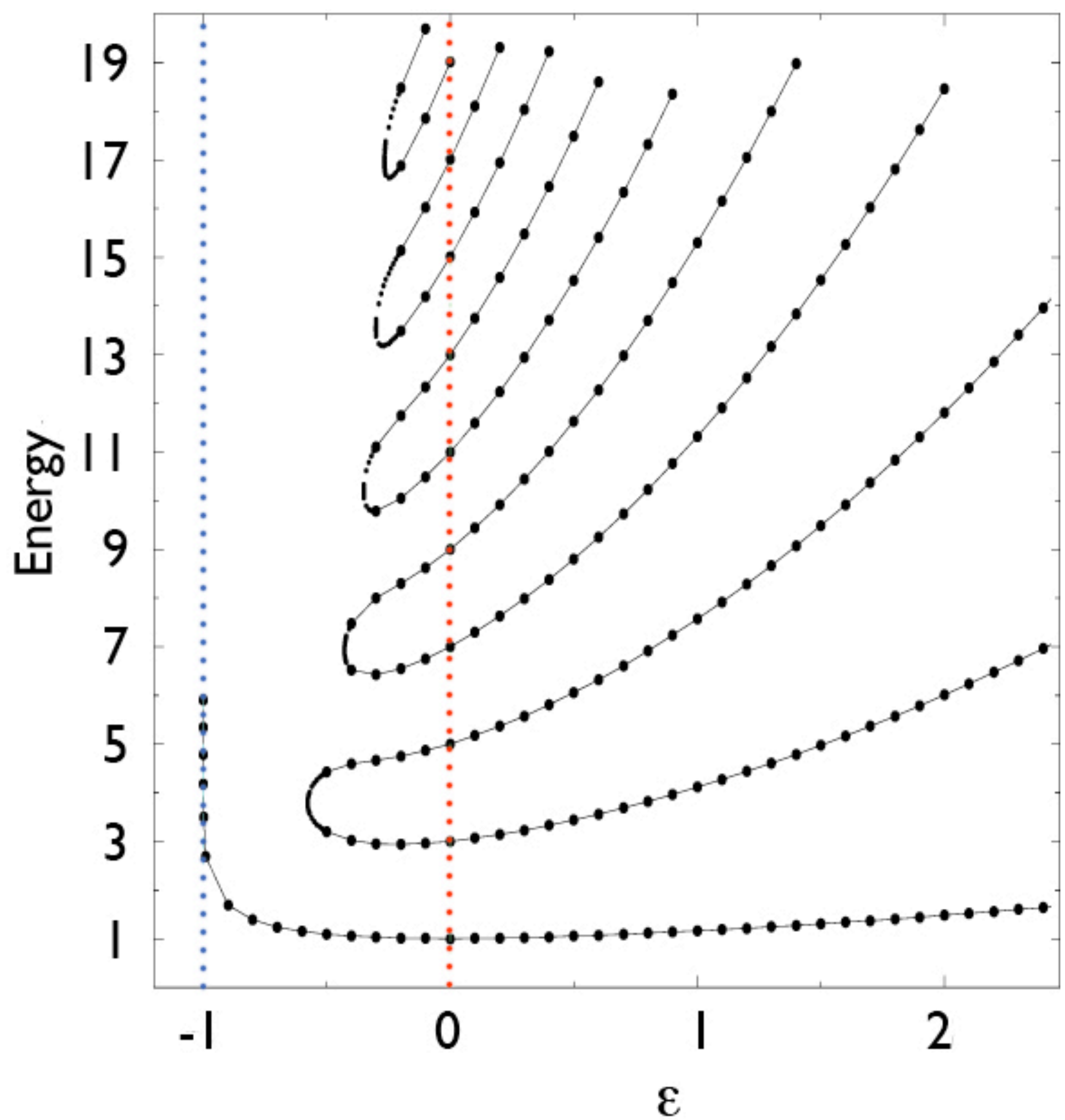
- convert into a differential equation form by replacing p by -*i*d/dx

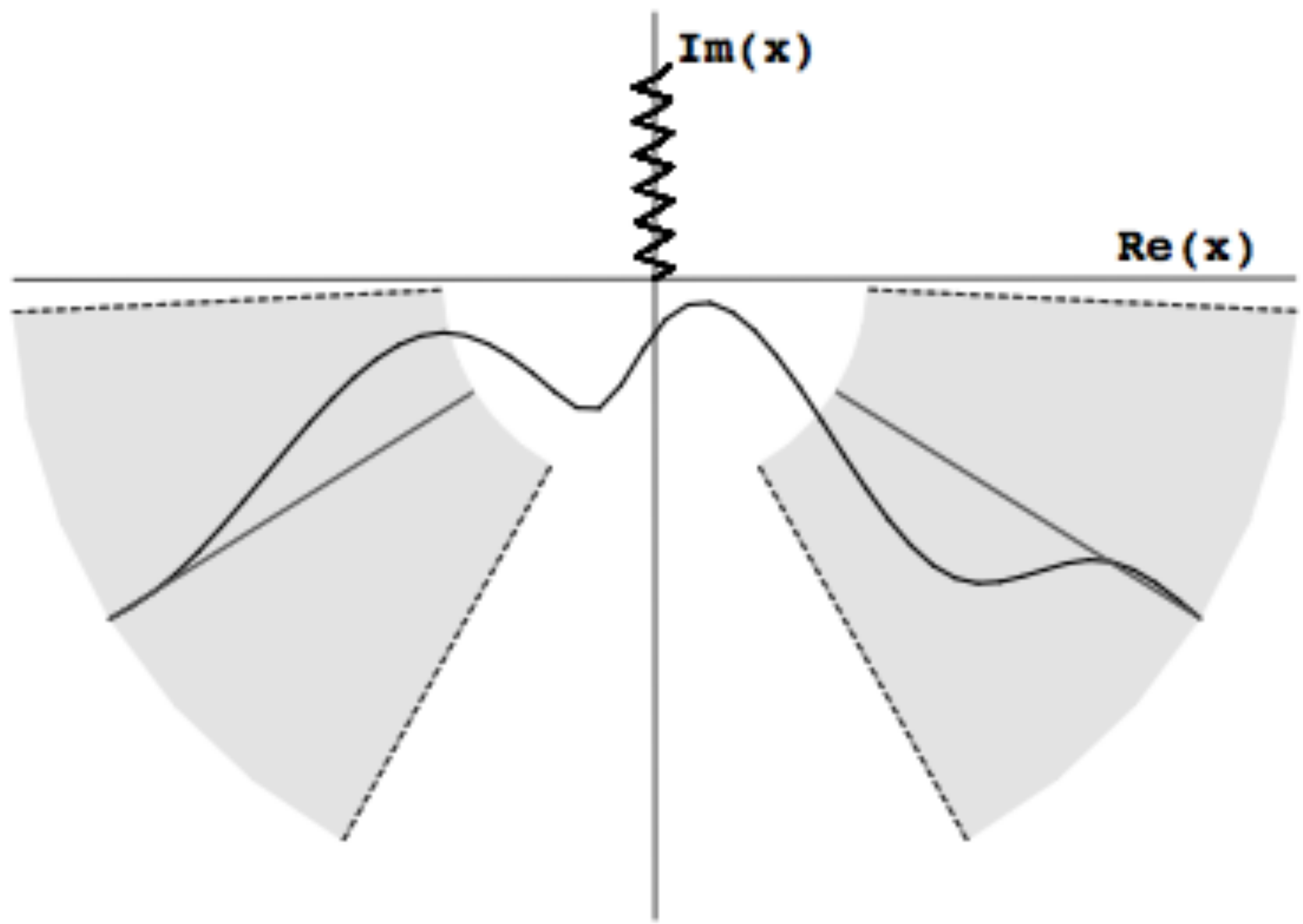
$$-\frac{d^2}{dx^2}\psi(x) + x^2(ix)^\epsilon\psi(x) = E\psi(x)$$

- Solve the eigenvalue problem numerically.

We are all familiar with the solution of the eigenvalue problem:

$$-\frac{d^2}{dx^2}\psi(x) + x^2(ix)^\epsilon\psi(x) = E\psi(x)$$



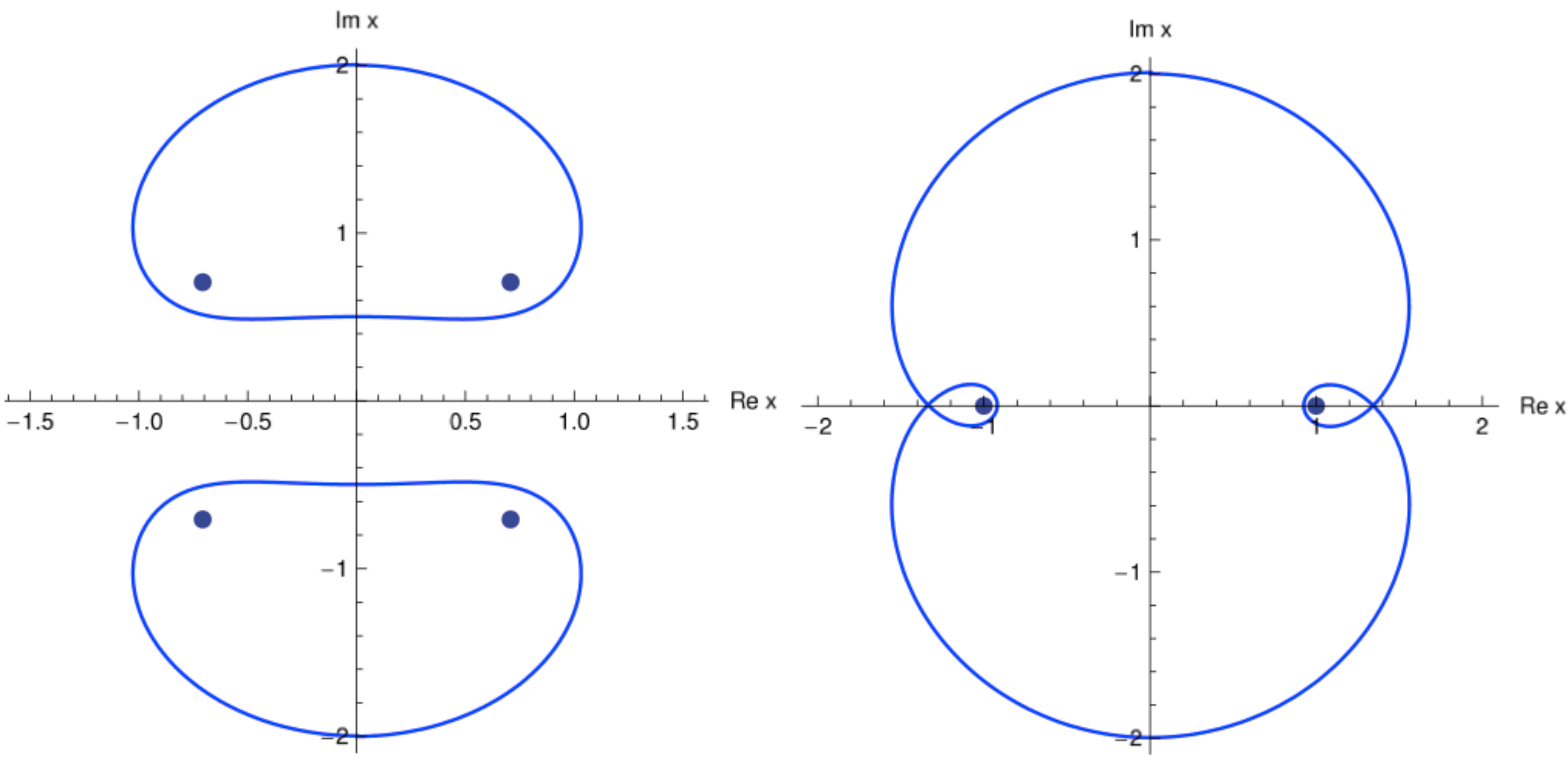

Im(x)
Re(x)

Isospectral Pairs of Hamiltonians

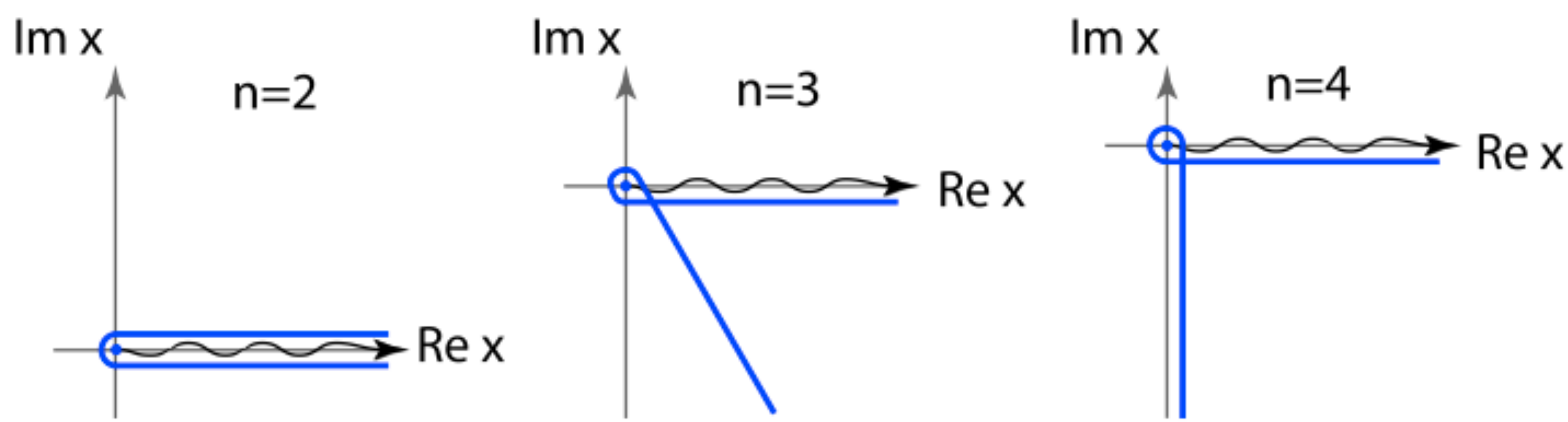
$$H_n = \eta p^n - \gamma(\mathrm{i}x)^{n^2}$$

For example:

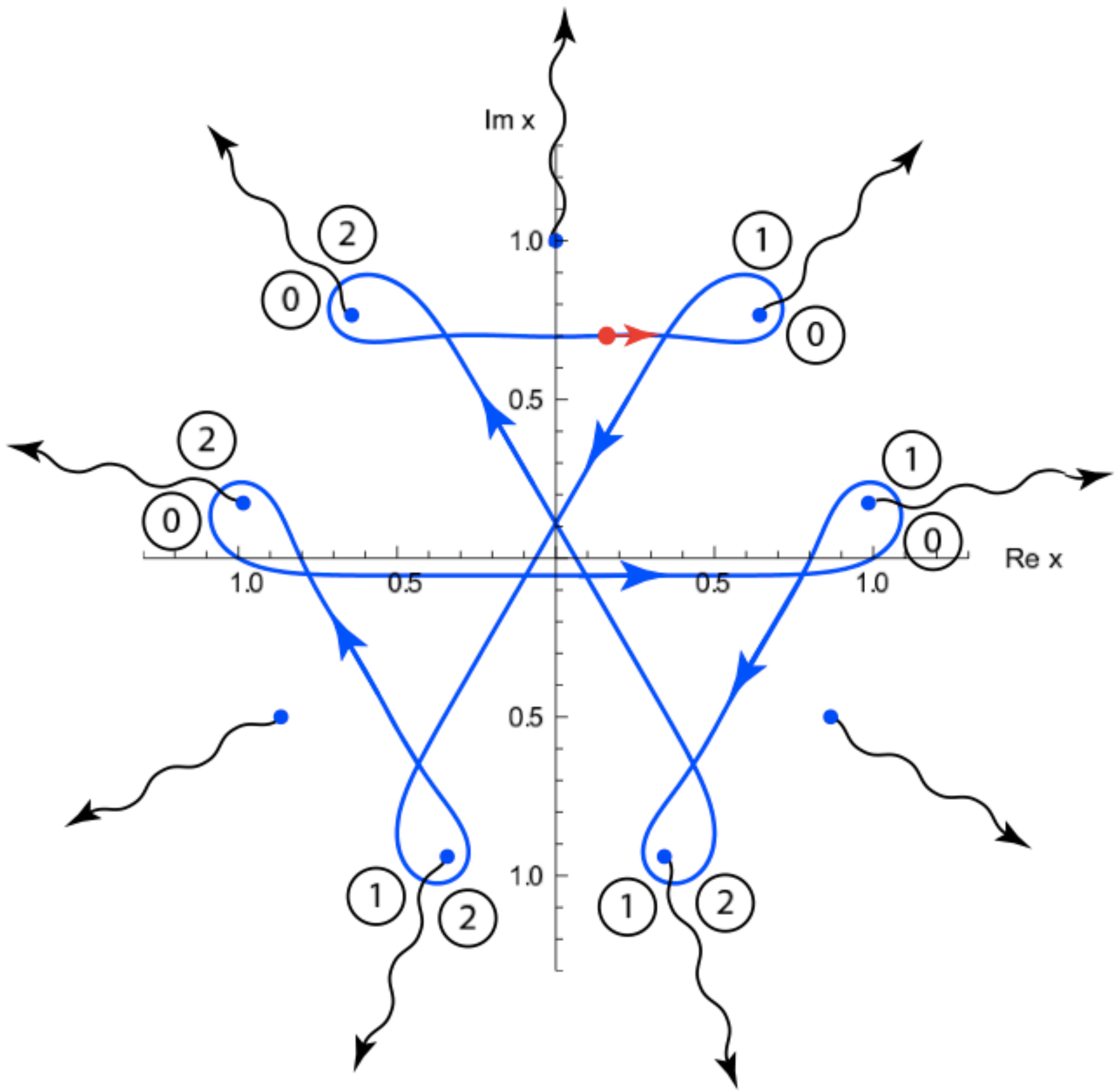
$$H_2 = \frac{1}{2m}p^2 - \gamma x^4 \qquad K_2 = \frac{1}{2m}x^2 + 4\gamma p^4 + \hbar\sqrt{\frac{2\gamma}{m}}p$$



Im x
Re x
−1.5
−1.0
−0.5
0.5
1.0
1.5
2
1
−1
−2
Im x
Re x
−2
−1
1
2
2
1
−1
−2



Im x
n=2
Re x
Im x
n=3
Re x
Im x
n=4
Re x

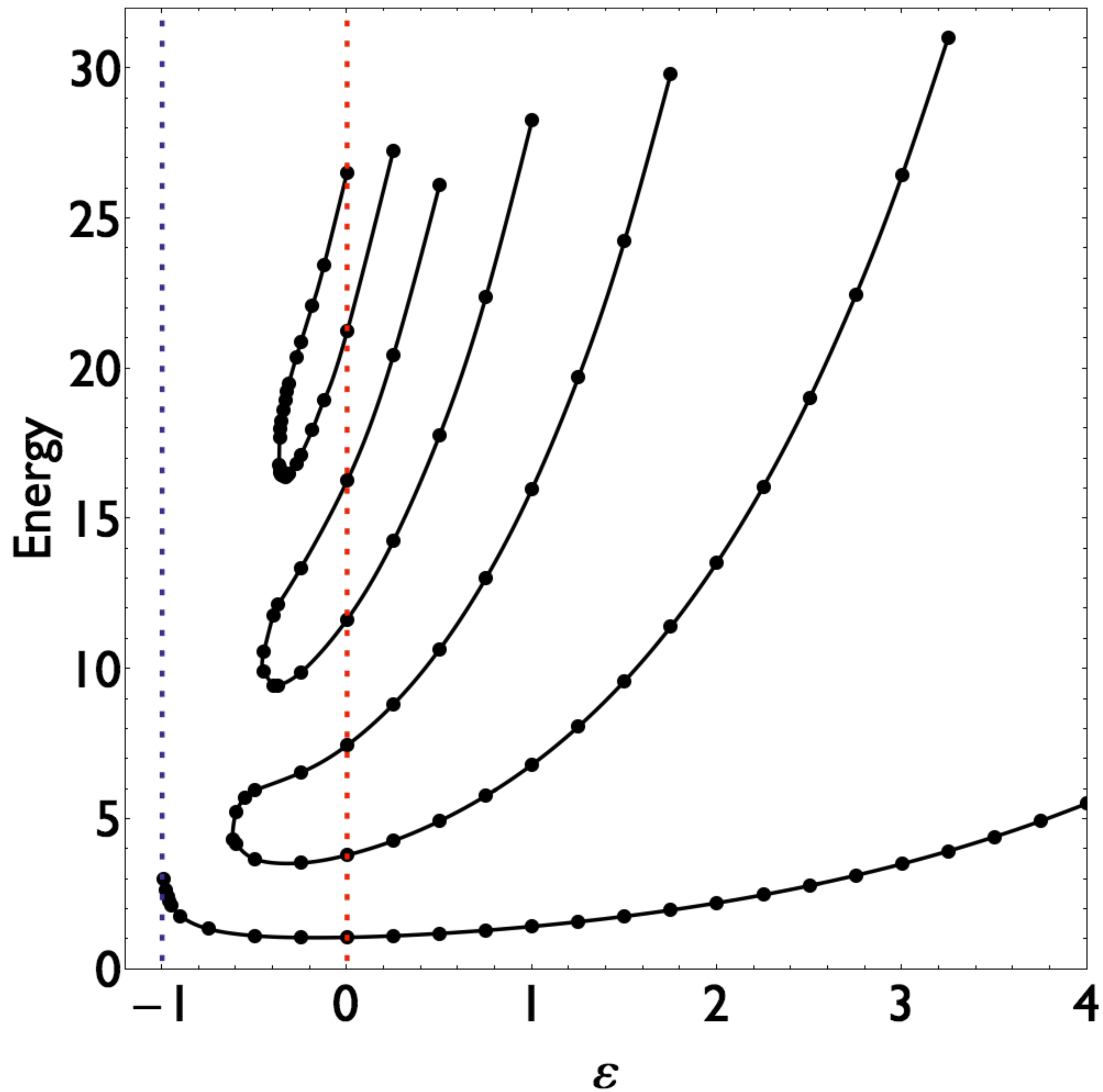

Im x
Re x
1.0
0.5
0.5
1.0
1.0
0.5
0.5
1.0
0
1
2

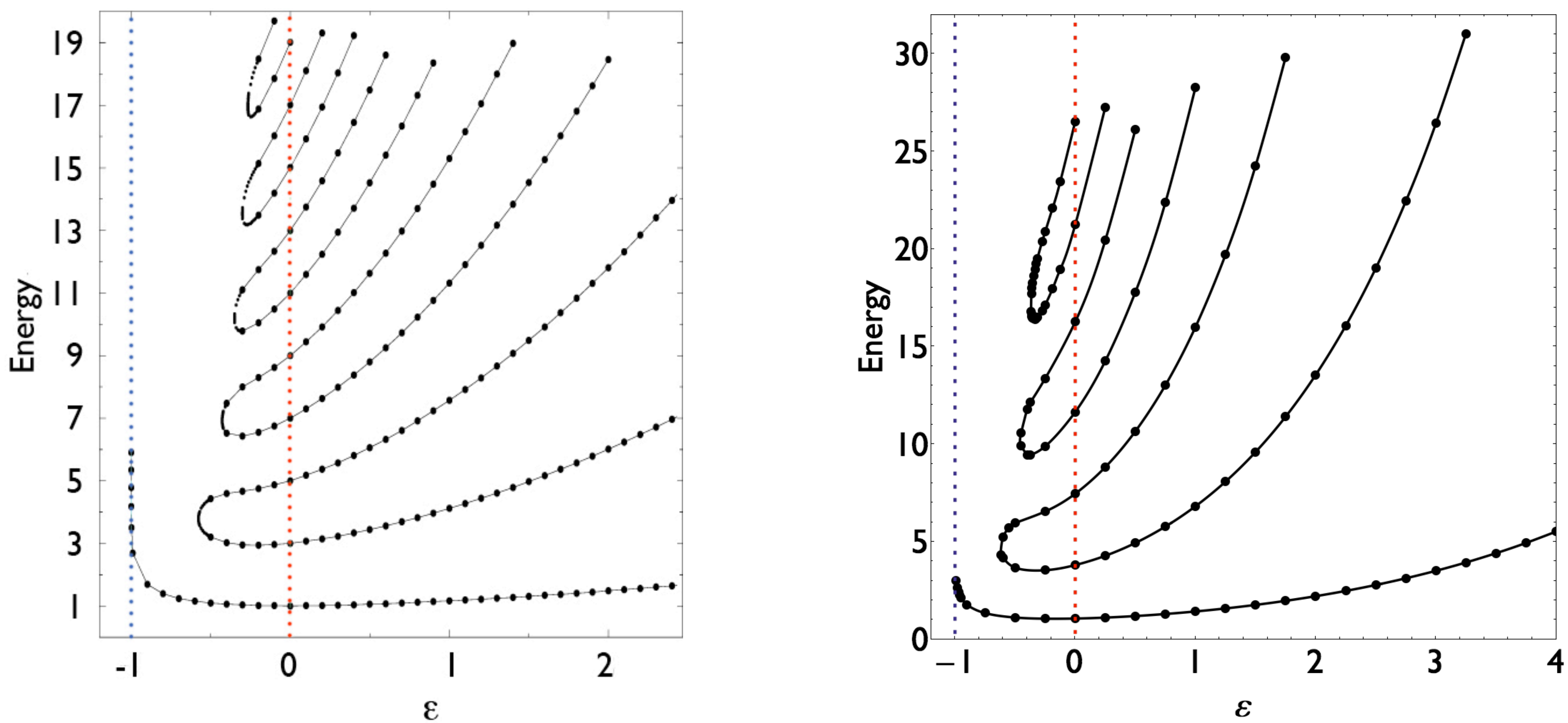
Isospectral Pairs of Hamiltonians

- We've already seen 4th order Hamiltonians...
- We know some of the qualities of the classical paths
- So...can we construct a general class of fourth order oscillators that have a real eigenspectrum? (i.e. is PT general or is it special to second order problems?)

$$\left(\frac{d}{dx}\right)^4 \psi(x) = E\psi(x) - x^2(\mathrm{i}x)^\epsilon \psi(x)$$

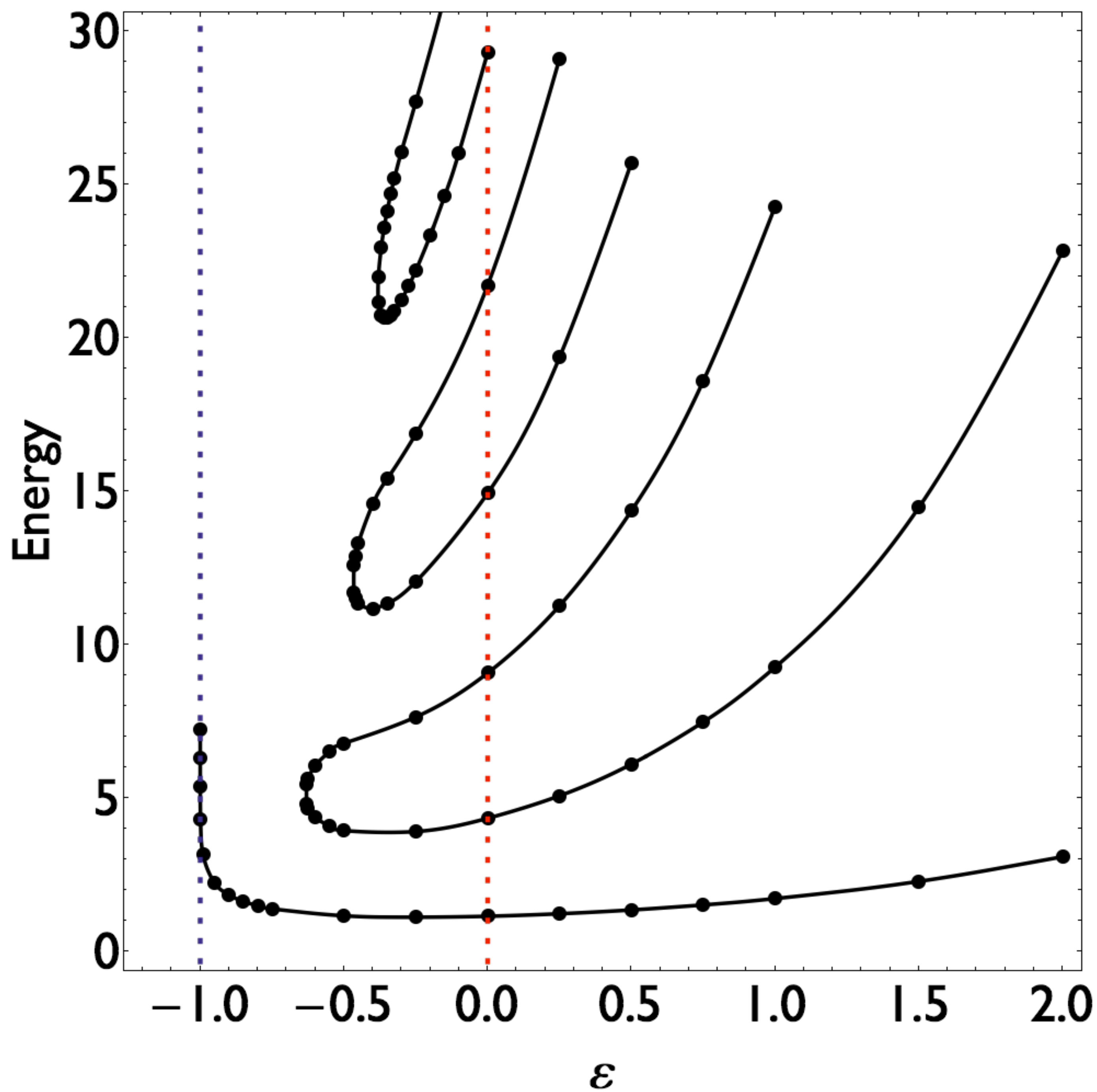
$$H = p^4 + x^2(\mathrm{i}x)^\epsilon$$





Energy
19
17
15
13
11
9
7
5
3
1
-1
0
1
2
ε
Energy
30
25
20
15
10
5
0
−1
0
1
2
3
4
ε

$$H = p^6 + x^2(\mathrm{i}x)^\epsilon$$



Worked Example

Four possible asymptotic behaviours:

$$\psi(x) \sim \exp\left(\frac{4\omega}{6+\varepsilon} x^{(6+\varepsilon)/4} e^{i\pi\varepsilon/8}\right) \quad (|x| \to \infty)$$

$$\omega^4 = -1$$

Stokes' wedges

Ignore "sign" so that the phase angle is: $(6+\varepsilon)\theta/4 + \pi\varepsilon/8$

Centre of wedge: $-\pi\varepsilon/(12+2\varepsilon)$

Angular opening: $4\pi/(6+\varepsilon)$

Worked Example

At $\varepsilon = 0$:

Opening angle is $2\pi/3$

As $\varepsilon \to \infty$:

Opening angle vanishes.

At $\varepsilon = -1$:

$\theta = \pi/10$ and $\Delta\theta/2 = 4\pi/10$

Thus, the upper edge of the right wedge is at $\pi/2$.

Numerical Procedure

Substitute into differential equation:

$$\psi_R^{(4)}(r) = -\left(r^2 + Ee^{4i\theta_R}\right)\psi_R(r) = -Q(r)\psi_R(r)$$

Calculate WKB, so that we have initial conditions:

$$\psi_{\mathrm{WKB}}\left(r_0\right) \sim KQ^{-\frac{3}{8}}\left(r_0\right)\exp\left[\omega\int_0^{r_0} dr\, Q^{1/4}(r)\right]$$

Choose two solutions:

$$\omega_1 = e^{3\pi i/4} \qquad \text{and} \qquad \omega_2 = e^{-3\pi/4}$$

Numerical Procedure

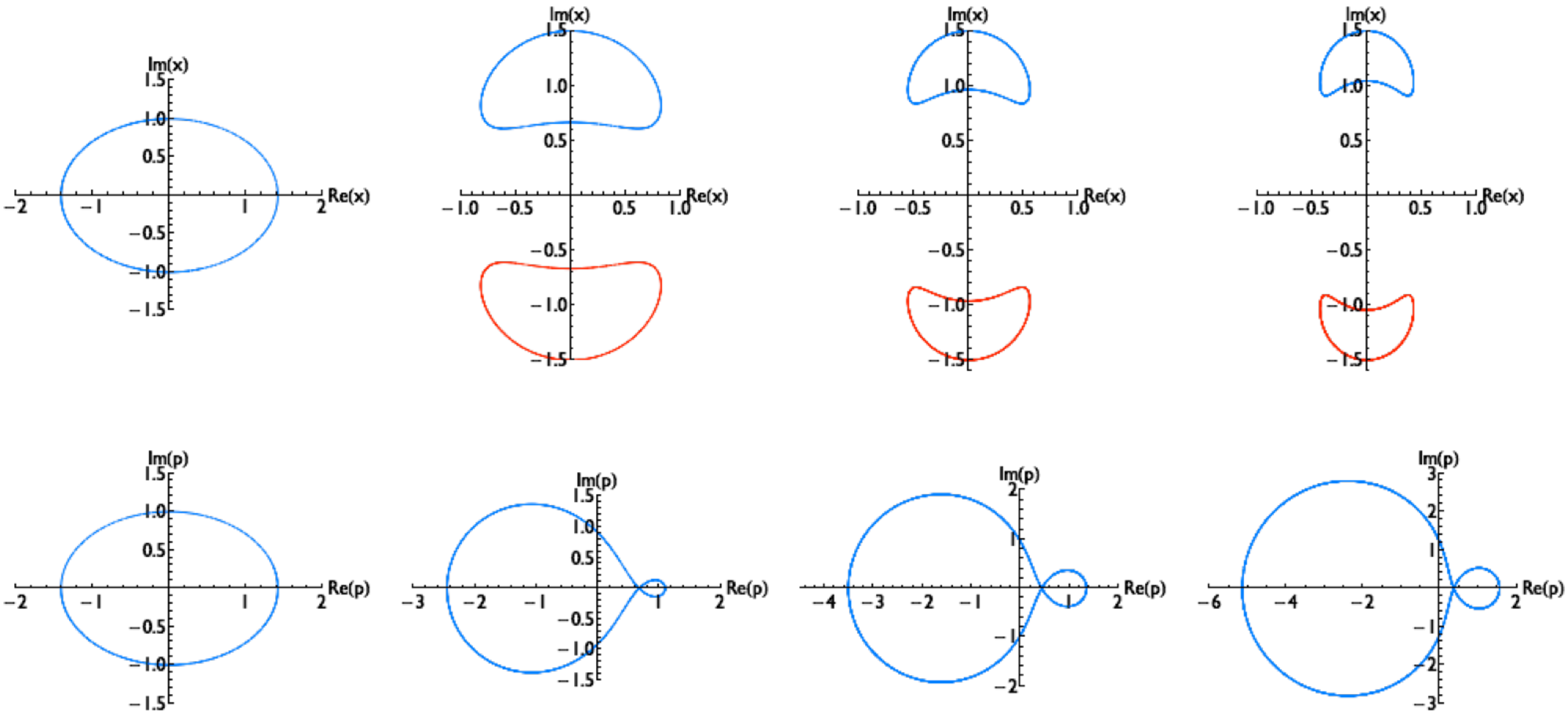
Require continuity:

$$\begin{aligned}
\alpha_1\psi_{L,1}(0) + \alpha_2\psi_{L,2}(0) &= \beta_1\psi_{R,1}(0) + \beta_2\psi_{R,2}(0),\\
\alpha_1 e^{-i\theta_L}\psi'_{L,1}(0) + \alpha_2 e^{-i\theta_L}\psi'_{L,2}(0) &= \beta_1 e^{-i\theta_R}\psi'_{R,1}(0) + \beta_2 e^{-i\theta_R}\psi'_{R,2}(0),\\
\alpha_1 e^{-2i\theta_L}\psi''_{L,1}(0) + \alpha_2 e^{-2i\theta_L}\psi''_{L,2}(0) &= \beta_1 e^{-2i\theta_R}\psi''_{R,1}(0) + \beta_2 e^{-2i\theta_R}\psi''_{R,2}(0),\\
\alpha_1 e^{-3i\theta_L}\psi'''_{L,1}(0) + \alpha_2 e^{-3i\theta_L}\psi'''_{L,2}(0) &= \beta_1 e^{-3i\theta_R}\psi'''_{R,1}(0) + \beta_2 e^{-3i\theta_R}\psi'''_{R,2}(0).
\end{aligned}$$

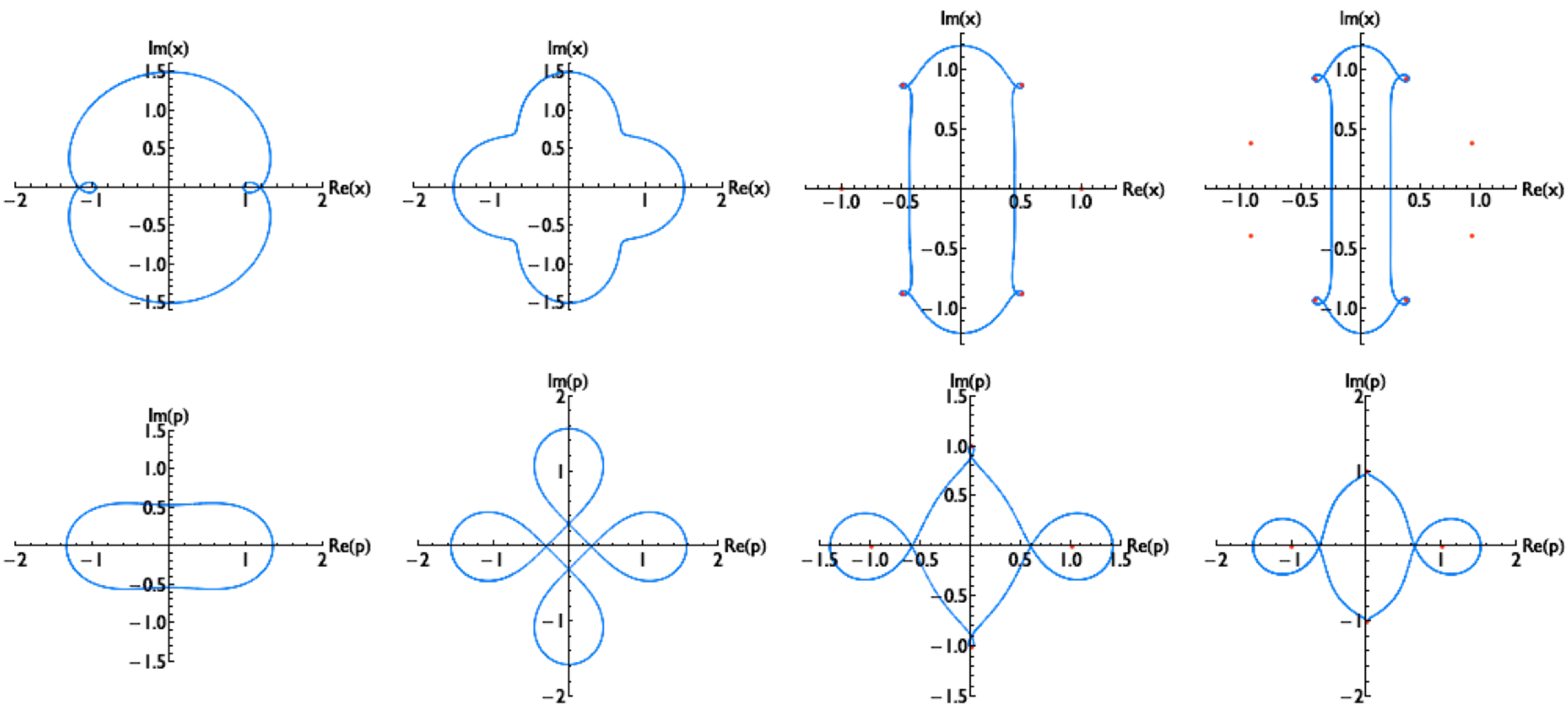
By Cramer's Rule, set Det = 0

$$\begin{pmatrix}
\psi_{R,1}(0) & \psi_{R,2}(0) & -\psi_{L,1}(0) & -\psi_{L,2}(0)\\
e^{-i\theta_R}\psi'_{R,1}(0) & e^{-i\theta_R}\psi'_{R,2}(0) & -e^{-i\theta_L}\psi'_{L,1}(0) & -e^{-i\theta_L}\psi'_{L,2}(0)\\
e^{-2i\theta_R}\psi''_{R,1}(0) & e^{-2i\theta_R}\psi''_{R,2}(0) & -e^{-2i\theta_L}\psi''_{L,1}(0) & -e^{-2i\theta_L}\psi''_{L,2}(0)\\
e^{-3i\theta_R}\psi'''_{R,1}(0) & e^{-3i\theta_R}\psi'''_{R,2}(0) & -e^{-3i\theta_L}\psi'''_{L,1}(0) & -e^{-3i\theta_L}\psi'''_{L,2}(0)
\end{pmatrix}$$

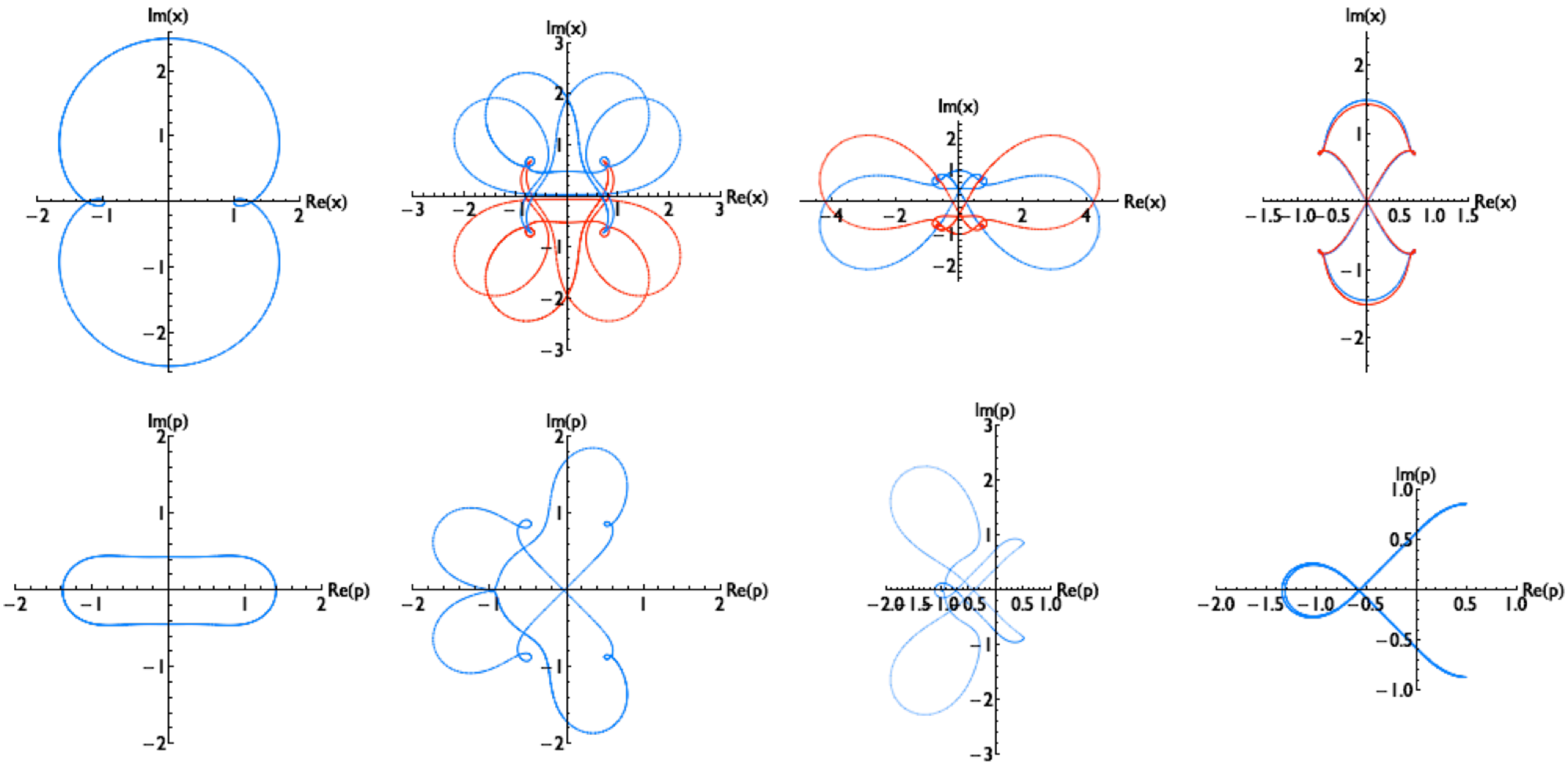
Classical Paths



Classical Paths



Classical Paths



Classical Paths