WKB for PT-symmetric Sturm-Liouville Problems

H. F. Jones

Imperial College London

PHHQP XI
Paris 27/08/12

Outline

Introduction

- The problem
- Two examples
- WKB

One-turning-point approximation

- igx
- ig sin(x)
- igx**3

Two-turning-point approximation

- igx**3
- igx**5
- Fractional Powers

Neumann Boundary Conditions

- 1TP
- 2TP

Summary

Outline

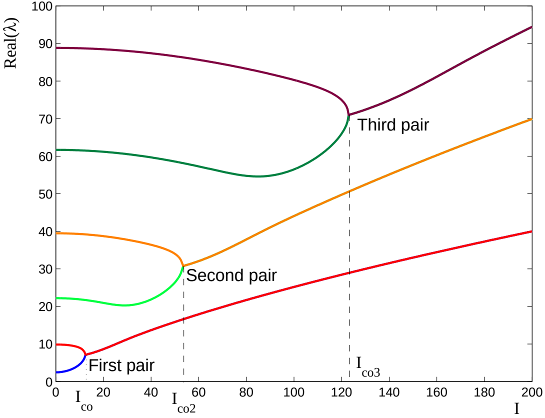
The problem

- Want to consider the PT-symmetric eigenvalue problem on a finite interval $[-L, L]$:

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = \lambda\psi$$

  with boundary conditions $\psi(\pm L) = 0$.
- Such problems, particularly with $V = \pm igx$, have physical significance.
- e.g. hydrodynamics (Günther et al.) →
  superconducting wire (Rubinstein et al.) →
  magnetic resonance signal (Zhao et al.) →

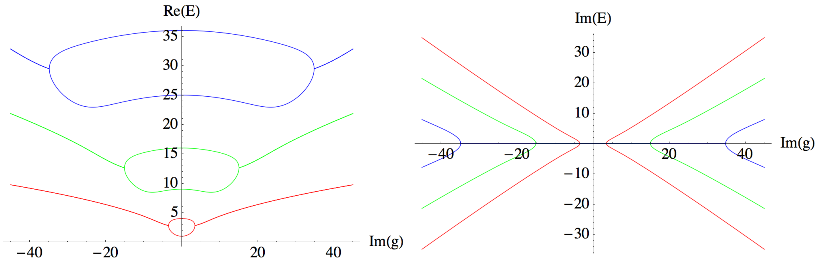
Example 1



Real parts of eigenvalues of $V = I(ix)$ on the interval $[-1, 1]$

Rubinstein et al., PRL **99**, 167003 (2007)

Example 2



Eigenvalues of $V = ig \sin x$ on the interval $[-\pi/2, \pi/2]$

Bender & Kalveks, Int. J. Theor. Phys. **50**, 955 (2011)

WKB

- Spectra are strikingly similar. Why?
  Maybe WKB approx. can cast some light.
- For large λ WKB solution of

$$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = \lambda\psi$$

  is

$$\psi(x) \sim \frac{1}{(\lambda - V(x))^{\frac{1}{4}}}\left(Ae^{i\int_{-L}^{x} ds\sqrt{\lambda - V(s)}} + Be^{-i\int_{-L}^{x} ds\sqrt{\lambda - V(s)}}\right)$$

- Imposing bd^y cond^{ns} at $x = \pm L$ gives quantization condition

$$\sin\left(\int_{-L}^{L} ds\sqrt{\lambda - V(s)}\right) = 0$$

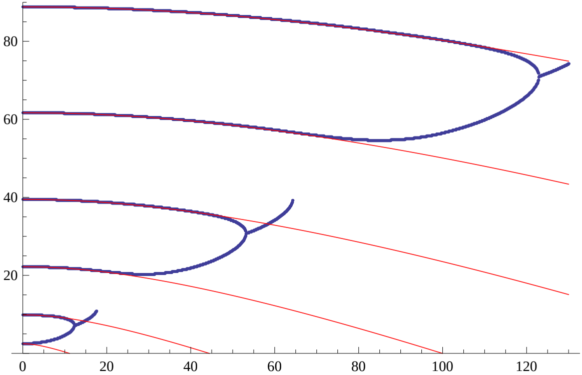
WKB

- i.e.

$$\int_{-L}^{L} ds \sqrt{\lambda_n - V(s)} = n\pi$$

- Path from $-L$ to L not specified, but assumed that it doesn't pass through a turning point (zero of $\lambda - V(s)$).
  Using this formula for $V = -igx$ gives following picture:

WKB



Simple WKB for $V = -igx$ vs. numerical results

WKB

- Clear that simple WKB doesn't give interesting structure

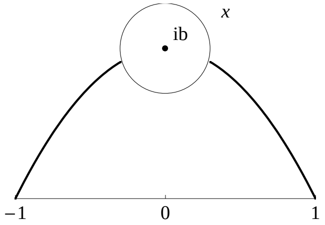
- Alternative? Take path through turning point(s)

CMB & HFJ, arXiv:1201.1234 [hep-th]

One turning-point

For $V = -igx$ write $\lambda - V(x)$ as $g(a + ix) \equiv gQ$

- $\exists$ single turning point at $x = ib \equiv ia$



1TP

Procedure: Away from turning-point use WKB approximations

$$\psi_L(x) \sim \frac{1}{[Q(x)]^{1/4}} \left\{ L_1 e^{i \int_x^{ib} ds\, \sqrt{gQ(s)}} + L_2 e^{-i \int_x^{ib} ds\, \sqrt{gQ(s)}} \right\}$$

on left-hand side, and

$$\psi_R(x) \sim \frac{1}{[Q(x)]^{1/4}} \left\{ R_1 e^{i \int_x^{ib} ds\, \sqrt{gQ(s)}} + R_2 e^{-i \int_x^{ib} ds\, \sqrt{gQ(s)}} \right\}$$

on the right.

1TP

- Then have to impose bd^y cond^{ns} $\psi(\pm 1) = 0$
  and match WKB approximations to Airy solutions

$$\psi_A = K_1 \mathrm{Ai}(y) + K_2 \mathrm{Ai}(\omega y)$$

  near TP.
  Here y is a scaled version of $x - ib$ such that the Schröd. eq^n becomes the Airy eq^n in the neighbourhood of *ib*,
  namely

$$y = g^{1/3}(x - ib)e^{-i\pi/6}$$

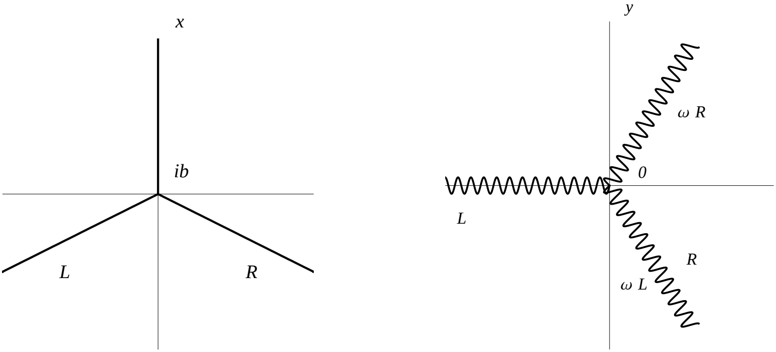
1TP

- Note that y can be considered large, since $y = g^{1/3}(x - ib)e^{-i\pi/6}$
- $\exists$ two different asymptotic approx^{ns} to Airy functions for y large:

$$
\begin{aligned}
\mathrm{Ai}(y) &\sim \frac{1}{2\sqrt{\pi}y^{1/4}} e^{-\frac{2}{3}y^{3/2}} \qquad (|\arg\, y| < \pi), \\
\mathrm{Ai}(y) &\sim \frac{1}{2\sqrt{\pi}y^{1/4}} \left(e^{-\frac{2}{3}y^{3/2}} + ie^{\frac{2}{3}y^{3/2}} \right) \qquad (|\arg\, y| = \pi).
\end{aligned}
$$

Which is appropriate depends on way you approach TP from L and R. Note that y-plane is rotated by $e^{-i\pi/6}$ w.r. to x-plane:

1TP



Relation between x- and y-planes near $x = ib$

1TP

- Have to further approximate WKB wave-functions for large y and match with Airy solution.
- On R just use simple approx. for both Airy functions.
- On L have to use second approx. for $\mathrm{Ai}(y)$.
- Altogether get 4 equations for 3 ratios L_1/L_2, R_1/R_2, K_1/K_2

  2 from $\psi(\pm 1) = 0$

  1 from matching on L

  1 from matching on R
- Hence (after much algebra!) get the eigenvalue condition

1TP

$$\boxed{\sin I_T + \frac{1}{2} e^{\Delta} = 0}$$

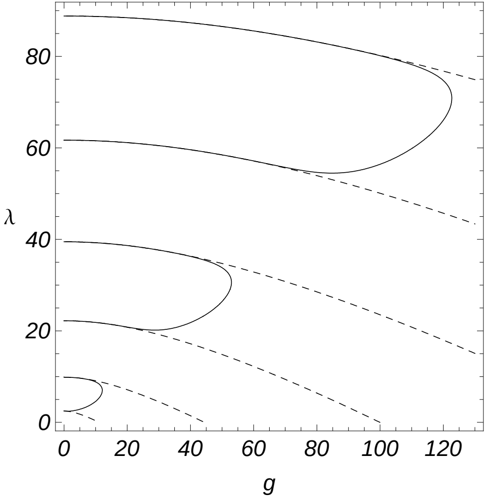
where

$$I_T = \int_{-1}^{1} ds \sqrt{gQ(s)}$$

$$\Delta \equiv 2\mathrm{Im} I_R$$

$$I_R = \int_{ib}^{1} ds \sqrt{gQ(s)}$$

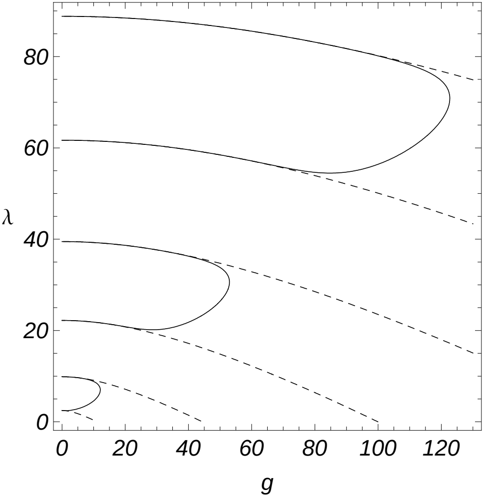
1TP



Simple and 1TP WKB for $V = -igx$

1TP

Essentially same story for $V = -ig \sin x$. Now $b = \arcsin a$.



Simple and 1TP WKB for $V = -ig \sin x$

1TP

Remarks:

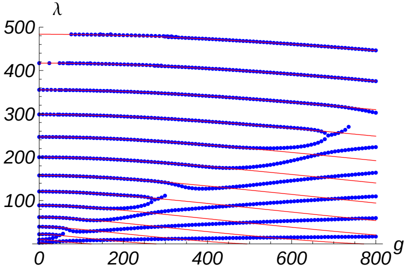
Recall 1TP eigenvalue condition

$$\sin I_T + \frac{1}{2} e^{\Delta} = 0$$

- First term by itself is 0TP approx^n
- $\Delta = 2\mathrm{Im} \int_{ib}^{1} ds \sqrt{gQ(s)}$ is generically large $\because \sqrt{g}$
- Second term negligible when $\Delta < 0$. Revert to 0TP approx^n
- No real solution when $\Delta > 0$
- Turn-around occurs near line ($\lambda = ag$) where $\Delta = 0$
- Mechanism for turn-around is cancellation between two terms

1TP

However, method fails when applied to $V = igx^3$:

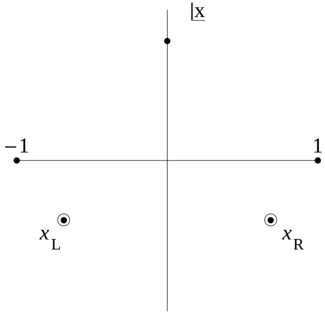


1TP WKB for $V = igx^3$

- Reason is that Δ is always < 0

1TP

But in this case there are three turning points:

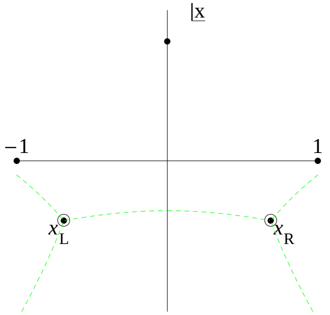


Turning points for $V = igx^3$

- Maybe correct path is through two lower turning points?

→ CMB & HFJ, arXiv:1203.5702[hep-th]

Two turning-points



Stokes lines for $V = igx^3$ with $a = 0.5$

2TP

- Now have different WKB approx^{ns} in three regions:

$$\left.\begin{array}{c}\psi_L(x)\\ \psi_M(x)\\ \psi_R(x)\end{array}\right\} \text{ in } \left\{\begin{array}{c}[-1, x_L]\\ [x_L, x_R]\\ [x_R, 1]\end{array}\right.$$

$$\begin{aligned}
\psi_L(x) &\sim \frac{1}{[Q(x)]^{1/4}}\left\{L_1 e^{i\int_x^{x_L} ds\,\sqrt{gQ(s)}} + L_2 e^{-i\int_x^{x_L} ds\,\sqrt{gQ(s)}}\right\}\\
\psi_M(x) &\sim \frac{1}{[Q(x)]^{1/4}}\left\{M_1 e^{i\int_{x_L}^{x} ds\,\sqrt{gQ(s)}} + M_2 e^{-i\int_{x_L}^{x} ds\,\sqrt{gQ(s)}}\right\}\\
\psi_L(x) &\sim \frac{1}{[Q(x)]^{1/4}}\left\{R_1 e^{i\int_{x_R}^{x} ds\,\sqrt{gQ(s)}} + R_2 e^{-i\int_{x_R}^{x} ds\,\sqrt{gQ(s)}}\right\}
\end{aligned}$$

- Won't need $\psi_R(x)$ explicitly: work only in LHP, and enforce PT symmetry of WF on imaginary axis.

2TP

PT symmetry $\Rightarrow \mathrm{Re}(\psi'(x)/\psi(x)) = 0$ on imag. axis

- Resulting condition is

$$\frac{M_1 M_2^*}{M_2 M_1^*} = e^{-2iI_M},$$

where $I_M \equiv \int_{x_L}^{x_R} ds \sqrt{gQ(s)}$

- Condition that $\psi(-1) = 0$ gives

$$\frac{L_1}{L_2} = -e^{-2iI_L},$$

where $I_L \equiv \int_{-1}^{x_L} ds \sqrt{gQ(s)}$

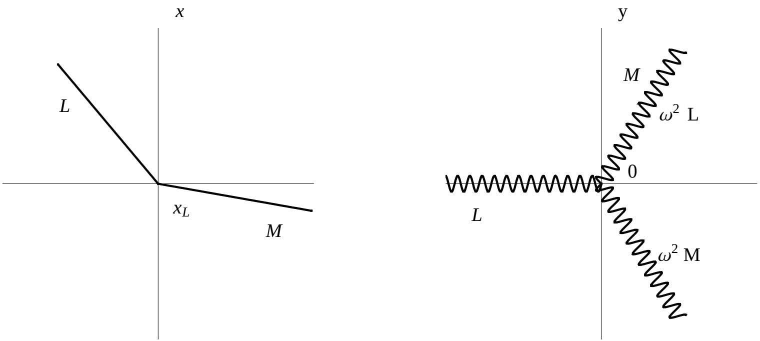
2TP

- Have to match ψ_L to Airy approx. in vicinity of x_L :
- In this case choose Airy expansion as

$$\psi_A(x) = K_1 \mathrm{Ai}(y) + K_2 \mathrm{Ai}(\omega^2 y),$$

  where $y = (x - x_L)/c$
- Now $c = \gamma e^{-i\theta/3}$, where $\theta = 5\pi/6$
- So matching paths are:

2TP



Relation between x- and y-planes near $x = x_L$

2TP

- Altogether get 4 equations for 3 ratios L_1/L_2, M_1/M_2, K_1/K_2
  - 1 from $\psi(-1) = 0$
  - 1 from PT-symmetry condition
  - 2 from matching on L and R of x_L
- Again get an eigenvalue condition, which in this case is

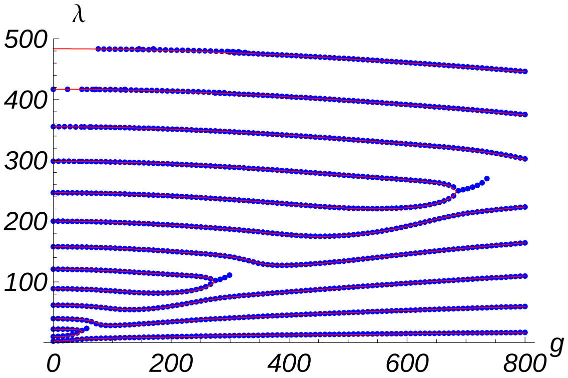
$$\boxed{\sin I_T + e^{\Delta_L} \cos I_M = 0}$$

where $\Delta_L \equiv 2\mathrm{Im} I_L$

NB: $\Delta_L \to -\Delta_L$ if TPs are in upper half plane

← cf. 1TP

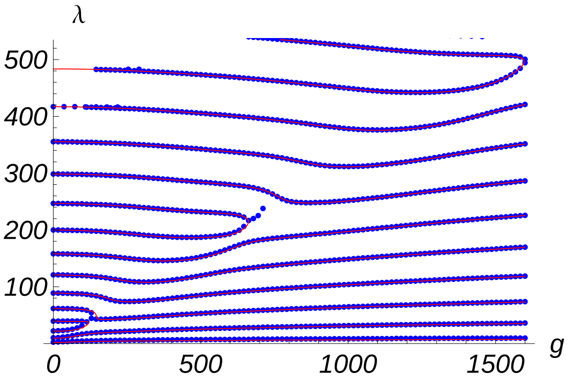
2TP



2TP WKB for $V = igx^3$ vs. numerical results

2TP

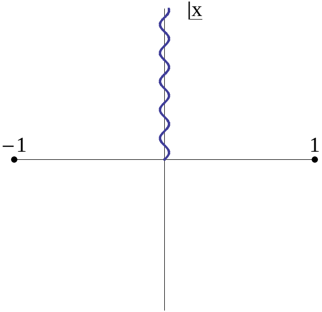
Same for $V = -igx^5$:



2TP WKB for $V = -igx^5$ vs. numerical results

2TP - Fractional Powers

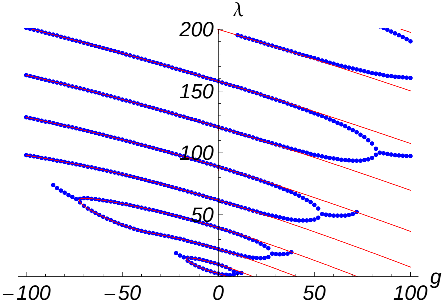
Now $\exists$ a cut in x-plane, which we take along $+$ve imaginary axis:



Cut x-plane for fractional powers in $V(x)$

2TP: $V = -g(ix)^{\frac{1}{2}}$

Spectrum no longer symmetric in g:

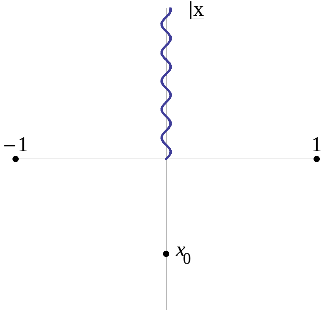


WKB for $V = -g(ix^{\frac{1}{2}})$ vs. numerical results:

- 1TP for $g < 0$ ✓
- 0TP for $g > 0$ ×

2TP: $V = -g(ix)^{\frac{1}{2}}$

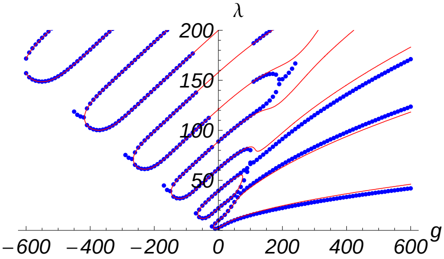
- $\exists$ only one TP at $x_0 = -ia^2$:



TP and cut for $V = -g(ix^{\frac{1}{2}})$

- TP on 1st sheet for $g < 0$ ✓
- TP on 2nd sheet for $g > 0$ ×

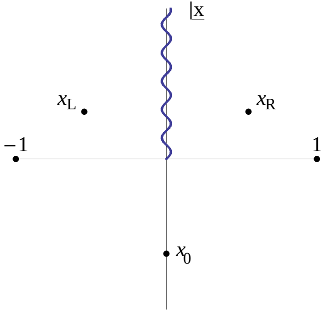
2TP: $V = -g(ix)^{3/2}$



WKB for $V = -g(ix^{3/2})$ vs. numerical results:

- 1TP for $g < 0$ ✓
- 2TP for $g > 0$ ×

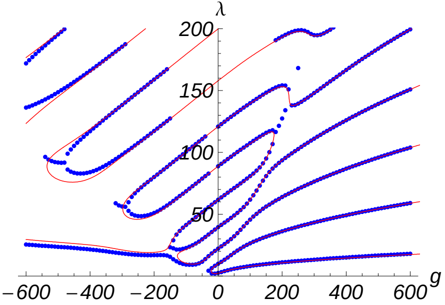
2TP: $V = -g(ix)^{3/2}$



TPs and cut for $V = -g(ix^{3/2})$

1TP for $g < 0$ ✓
2TP for $g > 0$ ×

2TP: $V = -g(ix)^{5/2}$

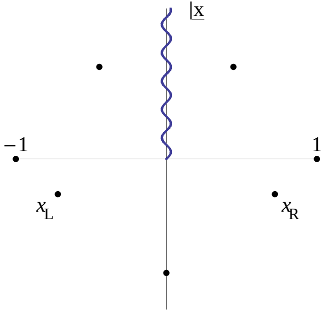


WKB for $V = -g(ix^{5/2})$ vs. numerical results:

Lower 2TP for $g > 0$ ✓

Upper 2TP for $g < 0$ ×

2TP: $V = -g(ix)^{5/2}$



TPs and cut for $V = -g(ix^{5/2})$

Lower 2TP for $g > 0$ ✓
Upper 2TP for $g < 0$ ×

Neumann Boundary Conditions

- Boundary conditions are now $\psi'(\pm 1) = 0$ rather than $\psi(\pm 1) = 0$
- Hence expressions for L_1/L_2 and R_1/R_2 change sign
- Net result is that sign of extra e^{Δ} term is **reversed**
- Note that $\exists$ new ground state with $\lambda = 0$ at $g = 0$, corresponding to $\psi = const$.

1TP - Neumann

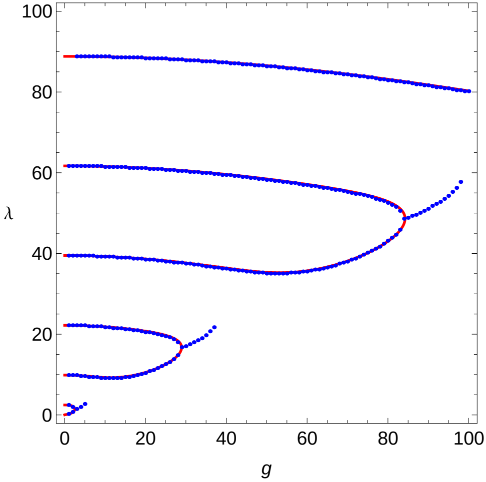
Eigenvalue equation is now

$$\boxed{\sin l_T - \tfrac{1}{2} e^{\Delta} = 0}$$

Gives different pairing:

igx - Neumann

← cf. Dirichlet



A scaled version of this graph occurs in Zhao et al.

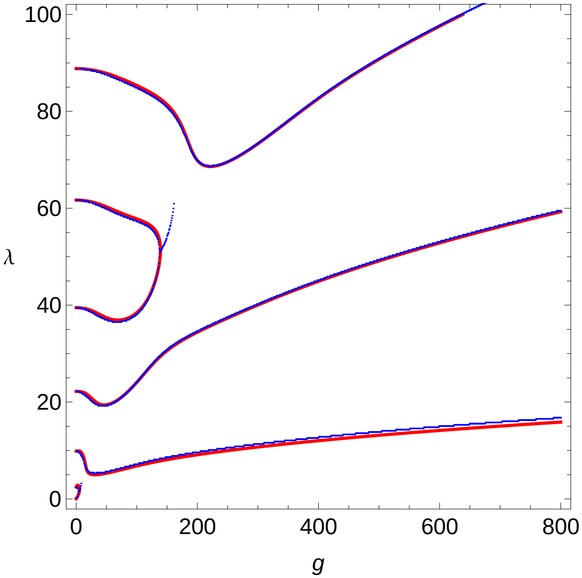
2TP - Neumann

Eigenvalue equation is now

$$\boxed{\sin l_T - e^{\Delta_L} \cos l_M = 0}$$

igx^3 - Neumann

← cf. Dirichlet



Summary

- 1TP approx^{n} works extremely well for $V = igx$, $V = i\sin x$ etc.
- Fails for $V = igx^3$. Need 2TP approx^{n}.
  2TP approx^{n} works well for $V = igx^3$, igx^5 etc.
- For fractional powers spectrum not symmetric in g .
  Cut $\Rightarrow$ WKB works for only one sign
- WKB formulae extremely simple. Exceptional points result from interaction of 1st and 2nd terms
- Interaction causes different pairings for Neumann boundary conditions

Summary

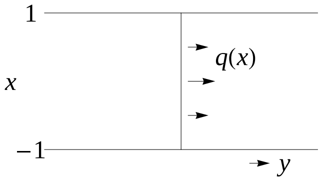
- WKB formulae very accurate, even for small g , λ
- General comment: breaking of PT symmetry for large g seems generic.
  With hindsight, continuum case was exceptional.

Hydrodynamics

Approximation to Orr-Sommerfeld equation

$$-i\varepsilon\frac{d^2}{dx^2}\varphi + q(x)\varphi = \lambda\varphi \qquad \text{with } \varphi(\pm 1) = 0$$

Here $\varepsilon \propto$ viscosity
$q(x)$ is velocity profile
λ is spectral parameter
$\varphi(x) \propto$ small perturbation to stream function



Hydrodynamics

Multiplying by $-i/\varepsilon$, and identifying $g = 1/\varepsilon$, get

$$\left(-\frac{d^2}{dx^2} - igq(x)\right)\varphi = -ig\lambda\varphi$$

- So in our language $V = -igq(x)$, with $q(x)$ real
- For PT-invariance need $q(x) = x^{2M+1}$
- Simplest possibility is $q(x) = x \leftrightarrow$ Couette flow

Superconducting Wire

Time-dependent Landau-Ginzburg model

$$\psi_t + i\varphi\psi = \psi_{xx} + \Gamma\psi - |\psi|^2\psi$$

Here ψ is order parameter
$\varphi = -Ix$ is electric potential
$\Gamma \propto T_c - T$

- Linearize about $\psi = 0$ and write $\psi = e^{(\Gamma-\lambda)t}u(x)$
- Then get eq^{n}

$$-u_{xx} - ixIu = \lambda u$$

Diffusion of Spin-Polarized Atoms in an Inhomogeneous Magnetic Field

Torrey equation

$$\left[D\nabla^2 - i\omega_L(\mathbf{x})\right]\psi = \frac{\partial\psi}{\partial t}$$

Here ψ is $\propto$ transverse magnetization
ω_L is Larmor frequency

- Approximate $\omega_L = \omega_0 + az + bx^2$
- Look for eigenmodes $\psi(t) = \varphi_\beta e^{-\beta t}$
- Factorize $\varphi_\beta = \varphi_\ell(x)\varphi_m(y)\varphi_n(z)$
- Then get

$$\left(D\frac{d^2}{dz^2} - iaz + E_n\right)\varphi_n(z) = 0$$

Bd^y cond^n taken to be $\psi'(\pm L) = 0$