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Non-Hermitian Quantum Theory  
from  
Covariance  
and the  
Correspondence Principle"

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# Deriving Non-Hermitian Quantum Theory from Covariance and the Correspondence Principle

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As a starting point it is demonstrated on the basis of the non-Hermitian Klein-Gordon equation, how covariance induces a pair of time-dependent Schrödinger equations. These so-called retarded and advanced Schrödinger equations reduce for Hermitian Hamilton operators to the well known time-dependent Schrödinger equation and its Hermitian conjugate. It is also shown via the separation ansatz that the retarded and advanced Schrödinger equation share even for non-Hermitian Hamilton operators the same stationary Schrödinger equation. Time-dependent wave functions solving the retarded or advanced Schrödinger equation can be expanded in terms of a suitable set of eigenfunctions of the stationary Schrödinger equation being orthogonal under integration along some complex contour interconnecting two suitable anti-Stokes cones induced by the long-distance part of the interaction potential. For matrix Hamiltonians this orthonormal set of stationary eigenfunctions reduces to the right and left eigenbasis of the respective matrix Hamiltonians. A suitable combination of the time-dependent retarded and advanced Schrödinger equation leads to some continuity equation yielding in correspondence to classical mechanics in the complex spacial plane some probabilistic interpretation of our non-Hermition Quantum Theory. Throughout the presentation the formalism being easily generalized also to Quantum Field Theory is demonstrated and tested by applying it to a list of suitable problems.

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# Some origin of non-Hermiticity

Example:

Diagonalizable matrix - Hamiltonians

Hamiltonian

matrix of right eigen vectors

$$H X = X D$$

↓      ↓

diagonal  
matrix of  
eigenvalues

$$X^{-1} H = D X^{-1}$$

$D = \begin{pmatrix} E_1 & \dots \\ & \ddots & E_n \end{pmatrix}$

$$H = X D X^{-1}$$

Here are two sources of non-Hermiticity:

$$\left. \begin{array}{l} 1) D \neq D^* \\ 2) X^+ \neq X^{-1} \end{array} \right\} \Rightarrow H \neq H^+$$

## $2 \times 2$ matrix-Hamiltonian

$$H = X \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} X^{-1}$$

$$= X [\varepsilon \mathbb{1}_2 + i\gamma \sigma_3] X^{-1}$$

$$\varepsilon = \frac{E_1 + E_2}{2}, \quad \gamma = \frac{E_1 - E_2}{2i}$$

$$\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

PT-symmetry (broken):  $E_1 = E_2^* = \varepsilon + i\gamma$

$$\Rightarrow H = X \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^* \end{pmatrix} X^{-1}$$

Example: C.M. Bender  $\Rightarrow E_{1/2} = r \cos \theta \pm i \sqrt{(r \sin \theta)^2 + (is)^2}$

$$H = \begin{pmatrix} r e^{i\theta} & s \\ s & r e^{-i\theta} \end{pmatrix} = r \cos \theta \mathbb{1}_2 + i [r \sin \theta \sigma_3 - i s \sigma_1]$$

$$= \underbrace{r \cos \theta}_{\varepsilon} \mathbb{1}_2 + i \underbrace{\sqrt{(r \sin \theta)^2 + (is)^2}}_{\gamma} \cdot \underbrace{\frac{r \sin \theta \sigma_3 - i s \sigma_1}{\sqrt{(r \sin \theta)^2 + (is)^2}}}_{C}$$

$$= e^{-\frac{\alpha}{2} \sigma_2} [\varepsilon \mathbb{1}_2 + i \gamma \sigma_3] e^{\frac{\alpha}{2} \sigma_2}$$

$$= \varepsilon \mathbb{1}_2 + i \gamma [\cosh \alpha \sigma_3 - i \sinh \alpha \sigma_1]$$

# Non-Hermitian retarded and advanced Schrödinger-equations

retarded:

$$i\hbar \partial_t |\psi^{(+)}(t)\rangle = H |\psi^{(+)}(t)\rangle$$

$$i\hbar \partial_t \langle \psi^{(+)}(t) | = \langle \psi^{(+)}(t) | H$$

advanced:  $-i\hbar \partial_t |\psi^{(-)}(t)\rangle = H |\psi^{(-)}(t)\rangle$

$$-i\hbar \partial_t \langle \psi^{(-)}(t) | = \langle \psi^{(-)}(t) | H$$

Hermitian conjugation:  $\langle \dots | = |\dots\rangle^+$ ,  $|\dots\rangle = \langle \dots |^+$

$$\Rightarrow \mp i\hbar \partial_t \langle \psi^{(\pm)}(t) | = \langle \psi^{(\pm)}(t) | H^+$$

$$\mp i\hbar \partial_t |\psi^{(\pm)}(t)\rangle = H^+ |\psi^{(\pm)}(t)\rangle$$

Solutions for stationary matrix-Hamiltonian  
 $H = X D X^{-1}$ :

$$|\psi^{(\pm)}(t)\rangle = e^{\pm \frac{1}{i\hbar} H t} |\psi^{(\pm)}(0)\rangle = X e^{\pm \frac{1}{i\hbar} D t} X^{-1} |\psi^{(\pm)}(0)\rangle$$

$$\langle \psi^{(\pm)}(t) | = \langle \psi^{(\pm)}(0) | e^{\pm \frac{1}{i\hbar} H^+ t} = \langle \psi^{(\pm)}(0) | X e^{\pm \frac{1}{i\hbar} D^* t} X^+$$

$$\langle \psi^{(\pm)}(t) | = \langle \psi^{(\pm)}(0) | e^{\mp \frac{1}{i\hbar} H^+ t} = \langle \psi^{(\pm)}(0) | X^{+1} e^{\mp \frac{1}{i\hbar} D^* t} X^+$$

$$|\psi^{(\pm)}(t)\rangle = e^{\mp \frac{1}{i\hbar} H^+ t} |\psi^{(\pm)}(0)\rangle = X^{+1} e^{\mp \frac{1}{i\hbar} D^* t} X^+ |\psi^{(\pm)}(0)\rangle$$

These holds:

$$\langle \psi^{(\mp)}(t) | \psi^{(\pm)}(t) \rangle = \langle \psi^{(\mp)}(0) | \psi^{(\pm)}(0) \rangle$$

Recall: Differential equation 2nd order in time decomposes into two 1st order equations:

"Klein-Gordon-like equation"

$$(i\hbar\partial_t)^2 - H^2 |\psi(t)\rangle = 0$$

$$\Rightarrow (i\hbar\partial_t - H)(i\hbar\partial_t + H) |\psi(t)\rangle = 0$$

$$\Rightarrow \underbrace{(i\hbar\partial_t - H)}_{\text{"retarded Schrödinger equ."}} |\psi^{(+)}(t)\rangle = 0, \underbrace{(i\hbar\partial_t + H)}_{\text{"advanced Schröd. equ."}} |\psi^{(-)}(t)\rangle = 0$$

$$\Rightarrow |\psi(t)\rangle = |\psi^{(+)}(t)\rangle + |\psi^{(-)}(t)\rangle$$

$$= e^{\frac{1}{i\hbar} H t} |\psi^{(+)}(0)\rangle + e^{-\frac{1}{i\hbar} H t} |\psi^{(-)}(0)\rangle$$

$$= X e^{\frac{1}{i\hbar} D t} X^{-1} |\psi^{(+)}(0)\rangle$$

$$+ X e^{-\frac{1}{i\hbar} D t} X^{-1} |\psi^{(-)}(0)\rangle$$

Analogously:

$$\langle\langle \psi(t) | = \langle\langle \psi^{(+)}(t) | + \langle\langle \psi^{(-)}(t) |$$

$$= \langle\langle \psi^{(+)}(0) | e^{\frac{1}{i\hbar} H t} + \langle\langle \psi^{(-)}(0) | e^{-\frac{1}{i\hbar} H t}$$

$$= \langle\langle \psi^{(+)}(0) | X e^{\frac{1}{i\hbar} D t} X^{-1}$$

$$+ \langle\langle \psi^{(-)}(0) | X e^{-\frac{1}{i\hbar} D t} X^{-1}$$

- Recall:  $((i\hbar\partial_t)^2 - H^2) |\psi(t)\rangle = 0$   
 $\Rightarrow (i\hbar\partial_t - H)(i\hbar\partial_t + H) |\psi(t)\rangle = 0$   
 $\Rightarrow i\hbar\partial_t |\psi^{(\pm)}(t)\rangle = \pm H |\psi^{(\pm)}(t)\rangle$   
 $i\hbar\partial_t \langle \psi^{(\pm)}(t) | = \pm \langle \psi^{(\pm)}(t) | H$

- Analogously: "Causal Klein-Gordon Equation"  
(Here 1-dim.)

$$\begin{aligned}
& ((i\hbar\partial_t)^2 - (-i\hbar c \frac{d}{dz})^2 - (m_0 c^2)^2) \psi(z, t) = 0 \\
\Rightarrow & (i\hbar\partial_t - \sqrt{(-i\hbar c \frac{d}{dz})^2 + (m_0 c^2)^2})(i\hbar\partial_t + \sqrt{(-i\hbar c \frac{d}{dz})^2 + (m_0 c^2)^2}) \psi(z, t) = 0 \\
\Rightarrow & i\hbar\partial_t \psi^{(\pm)}(z, t) = \pm \underbrace{\sqrt{(-i\hbar c \frac{d}{dz})^2 + (m_0 c^2)^2}}_{\approx m_0 c^2 + \frac{1}{2m_0} (-i\hbar \frac{d}{dz})^2} \psi^{(\pm)}(z, t)
\end{aligned}$$

- Hence we read off the retarded and advanced causal Schrödinger equations:

$$i\hbar\partial_t \psi^{(+)}(z, t) = + \left[ -\frac{t^2}{2m_0} \frac{d^2}{dz^2} + V(z) + m_0 c^2 \right] \psi^{(+)}(z, t)$$

$$i\hbar\partial_t \psi^{(-)}(z, t) = - \left[ -\frac{t^2}{2m_0} \frac{d^2}{dz^2} + V(z) + m_0 c^2 \right] \psi^{(-)}(z, t)$$

↑

Renormalizes  
the energy!

Starting point:

$$\pm i\hbar \partial_t \psi^{(\pm)}(z, t) = \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) \right] \psi^{(\pm)}(z, t)$$

Separation ansatz:  $\psi^{(\pm)}(z, t) = e^{\pm \frac{1}{i\hbar} Et} \phi_E(z)$

$$\Rightarrow \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) \right] \phi_E(z) = E \phi_E(z)$$

Construct orthogonal set of eigenfunctions!

$$\text{Recall: } \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) \right] \phi_n(z) = E_n \phi_n(z) \quad | \cdot \phi_m(z)$$

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) \right] \phi_m(z) = E_m \phi_m(z) \quad | \cdot \phi_n(z)$$

$$\Rightarrow -\frac{\hbar^2}{2m} \frac{d}{dz} \underbrace{\left[ \phi_n \frac{d}{dz} \phi_m - \phi_m \frac{d}{dz} \phi_n \right]}_{W[\phi_n, \phi_m]} = (E_n - E_m) \phi_n \cdot \phi_m$$

$$\Rightarrow -\frac{\hbar^2}{2m} \int_{z_0}^{z_1} dz \frac{d}{dz} W[\phi_n(z), \phi_m(z)] = (E_n - E_m) \int_{z_0}^{z_1} dz \phi_n(z) \phi_m(z)$$

$$\Rightarrow \text{If } W[\phi_n(z), \phi_m(z)] \Big|_{z_0}^{z_1} = 0 \Rightarrow \int_{z_0}^{z_1} dz \phi_n(z) \phi_m(z) \propto \delta_{nm}$$

$$\Rightarrow \psi^{(\pm)}(z, t) = \sum_n c_n^{(\pm)} e^{\pm \frac{1}{i\hbar} Ent} \phi_n(z)$$

$$\Rightarrow \int_{z_0}^{z_1} dz \psi^{(-)}(z, t) \psi^{(+)}(z, t) = \sum_n c_n^{(-)} c_n^{(+)} \int_{z_0}^{z_1} dz \phi_n(z)^2$$

$|z_0, z_1| \rightarrow \infty$  with  $z_0, z_1$  being  
in different Stokes-wedges

probability to be in  
state  $\phi_n$

# Schrödinger Equation in Holomorphic Representation

and

## Quantum Hamilton-Jacobi Theory

(= "modified de Broglie-Bohm approach")  
dBB

Start with:  $i\hbar \frac{d}{dt} \psi(z,t) = \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) \right] \psi(z,t)$

Assume:  $\boxed{z \in \mathbb{C}}$  !  $\rightarrow$  (causal) Holomorphic Representation

Separation & Eikonal Ansatz:  $\psi(z,t) = e^{\frac{i}{\hbar} S(z,t)} = \frac{e^{\frac{i}{\hbar} W(z) - Et}}{\psi(z) e^{\frac{i}{\hbar} Et}}$

### Quantum Hamilton-Jacobi Equation:

a) non-stationary:  $\frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial z} \right)^2 + V + \frac{i}{\hbar} \frac{1}{2m} \frac{\partial^2 S}{\partial z^2} = 0$

b) stationary:  $-E + \frac{1}{2m} \left( \frac{\partial W}{\partial z} \right)^2 + V + \frac{i}{\hbar} \frac{1}{2m} \frac{\partial^2 W}{\partial z^2} = 0$

Classical HJ-Equation      Quantum-Potential

~~"Riccati-equation"~~

Define:  $p(z,t) \equiv \frac{\partial S(z,t)}{\partial z} \stackrel{!}{=} \frac{\partial W(z)}{\partial z}$  ← Compare  
G.Wentzel, Z.Phys.  
38(1926)518

⇒ nonstationary:  $\frac{\partial S}{\partial t} + \frac{1}{2m} p^2 + V + \frac{i}{\hbar} \frac{1}{2m} \frac{\partial P}{\partial z} = 0$  !

b) stationary:  $-E + \frac{1}{2m} p^2 + V + \frac{i}{\hbar} \frac{1}{2m} \frac{\partial P}{\partial z} = 0$

Quantum-Potential  
is ~~real~~ complex!

⇒ Corresponding Hamilton-fct:

$$H = \frac{p^2}{2m} + V + \frac{i}{\hbar} \frac{1}{2m} \frac{\partial P}{\partial z}$$

⇒ Hamilton-equations:

$$\frac{dz}{dt} = \frac{\partial H}{\partial p} = \frac{P}{m}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z} - \frac{i}{\hbar} \frac{1}{2m} \frac{\partial^2 P}{\partial z^2}$$

Consider free 1-dimensional particle:

$$\psi(x) = A e^{i\sqrt{2mE/\hbar}x/\hbar} + B e^{-i\sqrt{2mE/\hbar}x/\hbar}, \quad A, B \in \mathbb{C}$$

$$\Rightarrow \frac{dx}{dt} = \sqrt{\frac{2E}{m}} \frac{\alpha e^{i\sqrt{2mE/\hbar}x/\hbar} - e^{-i\sqrt{2mE/\hbar}x/\hbar}}{\alpha e^{i\sqrt{2mE/\hbar}x/\hbar} + e^{-i\sqrt{2mE/\hbar}x/\hbar}}$$

$$\text{Define } \bar{x} = \frac{\sqrt{2mE}}{\hbar} x, \bar{t} = \frac{2E}{\hbar} t.$$

$$\Rightarrow \frac{d\bar{x}}{d\bar{t}} = \frac{\alpha e^{i\bar{x}} - e^{-i\bar{x}}}{\alpha e^{i\bar{x}} + e^{-i\bar{x}}}, \quad \bar{x}(\bar{t}_0) = \bar{x}_0 \in \mathbb{C}, \quad \alpha = \frac{A}{B}$$

$$\text{Initial conditions } \bar{x}_0, \dot{\bar{x}}_0 \Rightarrow \boxed{\alpha = \frac{1 + \dot{\bar{x}}_0}{1 - \dot{\bar{x}}_0} e^{-2i\bar{x}_0}}$$

Special cases:  $\alpha = 0 \Rightarrow$  left moving,  $\alpha = \infty \Rightarrow$  right moving

C.-D. Yang / Annals of Physics 319 (2005) 444–470

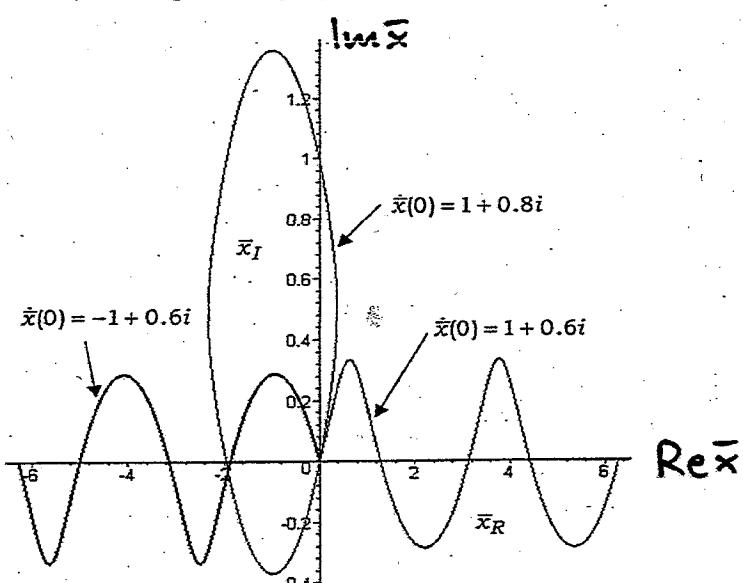


Fig. 1. Three types of trajectory for free particle with same initial position at the origin but with different initial velocities.

### Harmonic Oszillator:

$$\underline{n=0}: \psi_0 \propto e^{-\alpha^2 x^2/2} e^{-iE_0 t/\hbar} \quad (E_0 = \frac{\hbar\omega}{2})$$

$$\Rightarrow \dot{x} = -\frac{\hbar}{im} \alpha^2 x \Rightarrow x = A e^{it\alpha^2 t/m}$$

$$\underline{n=1}: \psi_1 \propto e^{-\alpha^2 x^2/2} 2\alpha x e^{-iE_1 t/\hbar}$$

$$\Rightarrow \dot{x} = \frac{\hbar}{im} \left( -\alpha^2 x + \frac{1}{x} \right) \Rightarrow (\alpha x - 1)(\alpha x + 1) = A e^{2it\alpha^2 t/m}$$

$$\Rightarrow x = \pm \frac{1}{\alpha} \sqrt{1 + A e^{2it\alpha^2 t/m}}$$

$$\underline{n=2}: \text{analogue} \Rightarrow 4\alpha x (\alpha x - \sqrt{\frac{5}{2}})^2 (\alpha x + \sqrt{\frac{5}{2}})^2 = A e^{5it\alpha^2 t/m}$$

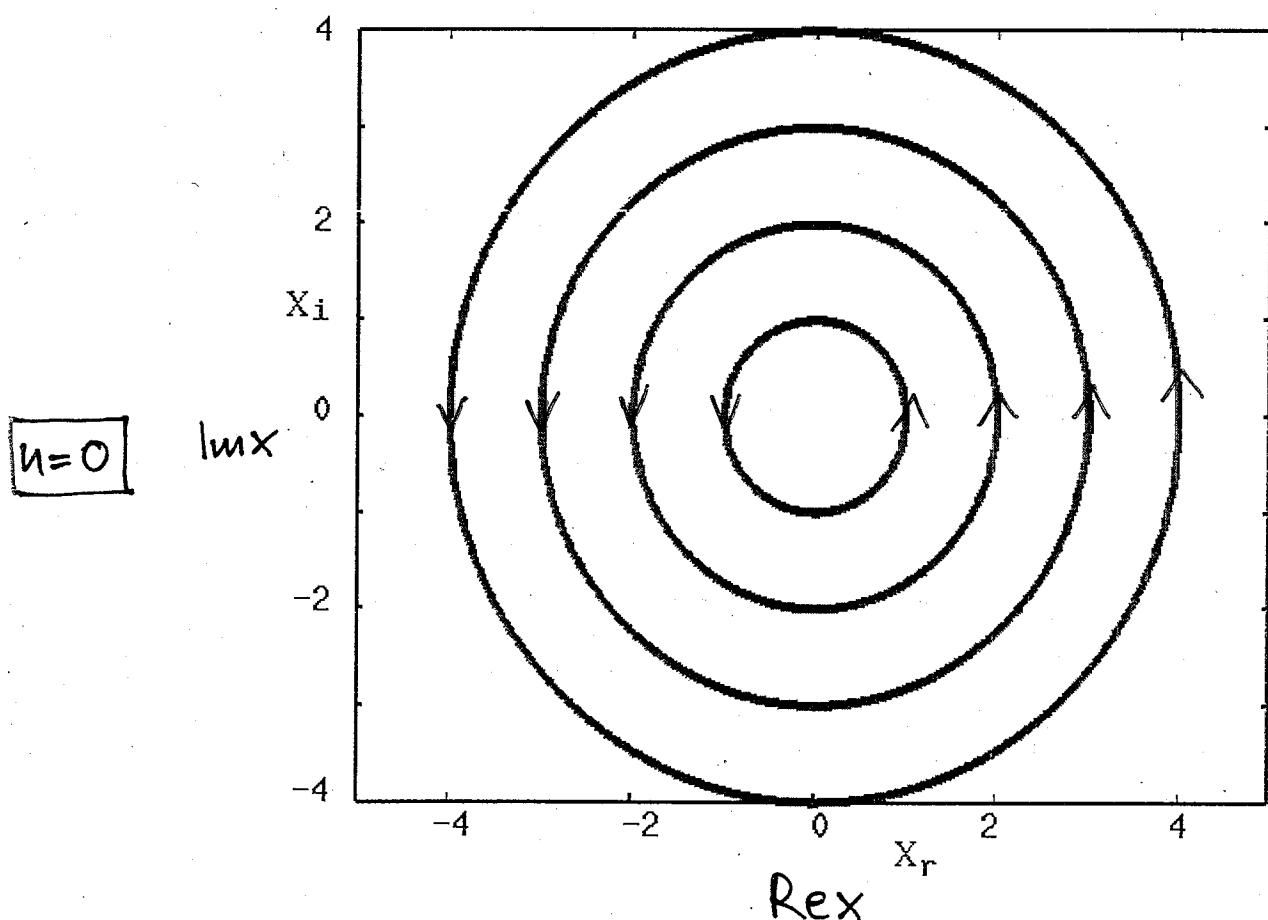


Figure 1: Complex paths in the  $X$ -plane ( $X \equiv \alpha x$ ) for the  $n = 0$  harmonic oscillator case, where contours are plotted for  $X(0) = 1, 2, 3$  and  $4$ .

Harmonic Oscillator  $n=1,2$

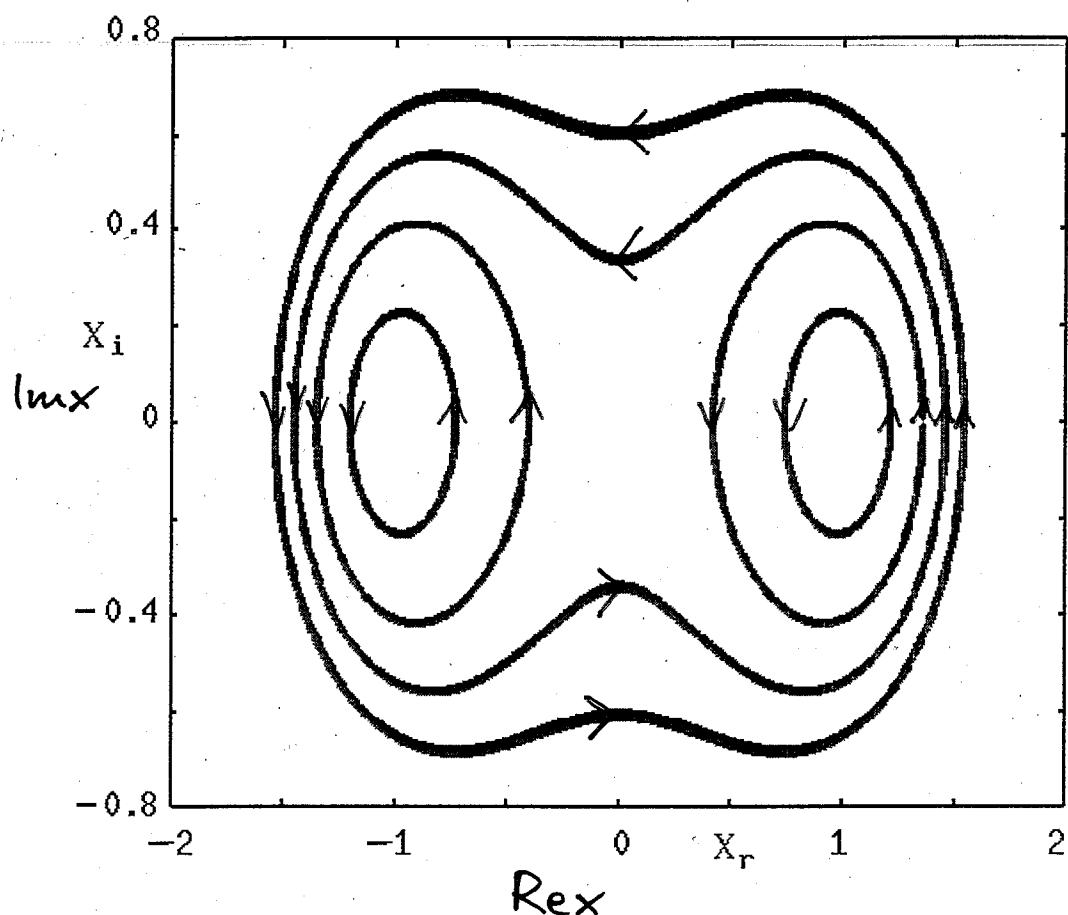


Figure 2: Complex paths in the  $X$ -plane ( $X \equiv \alpha x$ ) for the  $n = 1$  harmonic oscillator case, where contours are plotted for  $X(0) = 1.2, 1.35, 1.45$  and  $1.55$ .

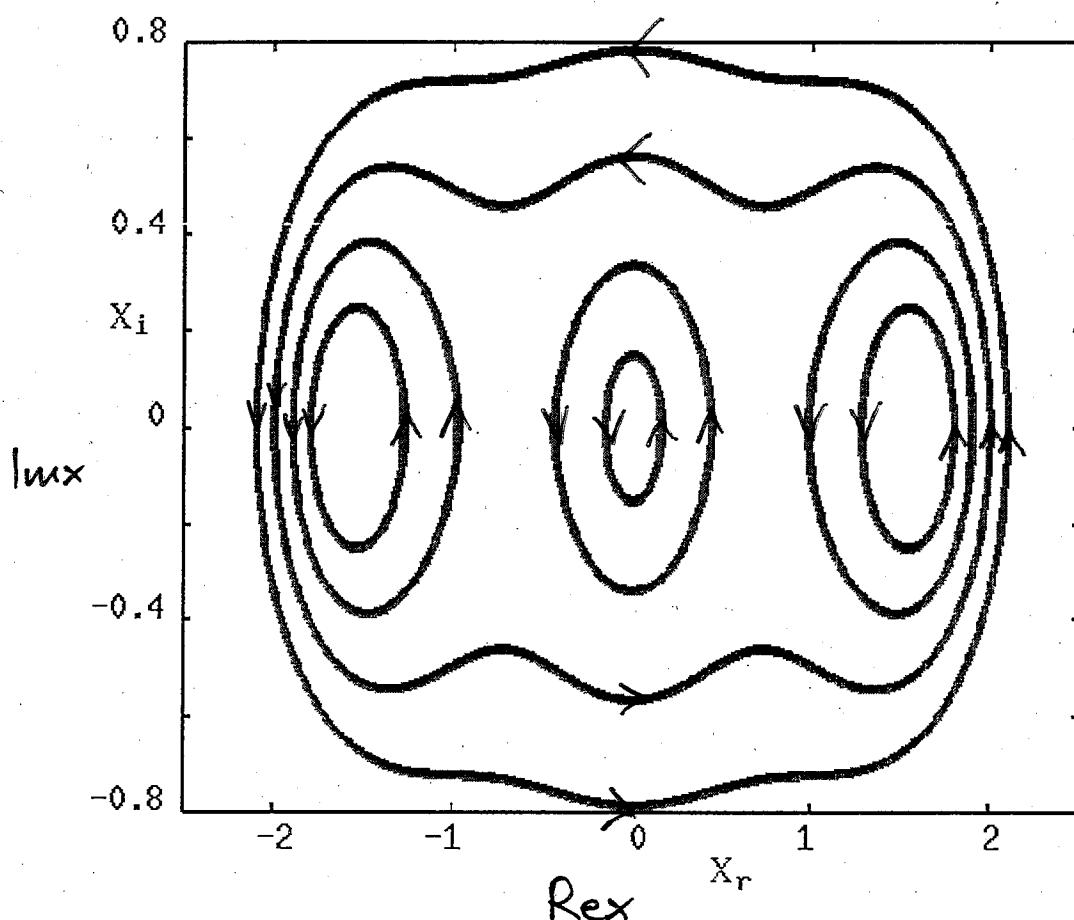


Figure 3: Complex paths in the  $X$ -plane ( $X \equiv \alpha x$ ) for the  $n = 2$  harmonic oscillator case, where contours are plotted for  $X(0) = 1.8, 1.9, 2.0$  and  $2.1$ .

C. M. Bender, S. Boettcher, P. Meisinger,  
*J. Math. Phys.* 40 (1999) 2201 (= quant-ph/  
 8803072)

$$H = p^2 - ix \quad (\epsilon = -1) \quad (3)$$

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \in \mathbb{R})$$

$$\Rightarrow \frac{1}{2} \frac{dx}{dt} = \pm \sqrt{E + (ix)^{2+\epsilon}}$$

rescale!

$$\frac{dx}{dt} = \pm \sqrt{1 + (ix)^{2+\epsilon}}$$

$$H = p^2 + x^2 \quad (\epsilon = 0)$$

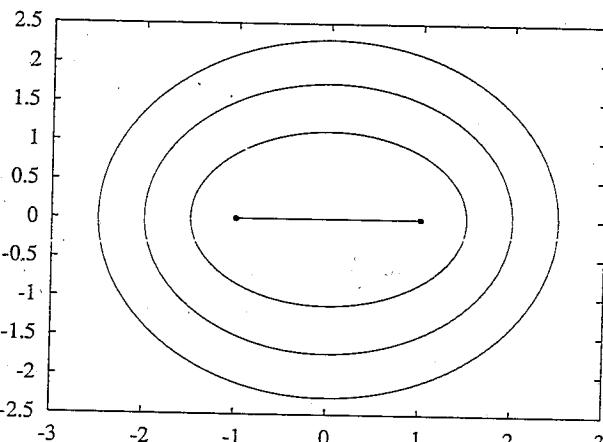


FIG. 1. Classical trajectories in the complex- $x$  plane for the harmonic oscillator whose Hamiltonian is  $H = p^2 + x^2$ . These trajectories represent the possible paths of a particle whose energy is  $E = 1$ . The trajectories are nested ellipses with foci located at the turning points at  $x = \pm 1$ . The real line segment (degenerate ellipse) connecting the turning points is the usual periodic classical solution to the harmonic oscillator. All closed paths [see Eq. (2.6)] have the same period  $2\pi$ .

$$H = p^2 - x^4 \quad (\epsilon = 2)$$

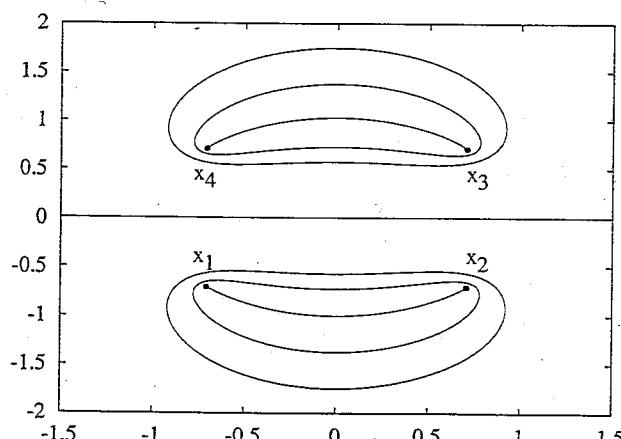


FIG. 4. Classical trajectories in the complex- $x$  plane for a particle described by the Hamiltonian  $H = p^2 - x^4$  and having energy  $E = 1$ . There are two oscillatory trajectories connecting the pairs of turning points  $x_1$  and  $x_2$  in the lower-half  $x$ -plane and  $x_3$  and  $x_4$  in the upper-half  $x$ -plane. [A trajectory joining any other pair of turning points is forbidden because it would violate  $\mathcal{PT}$  (left-right) symmetry.] The oscillatory trajectories are surrounded by closed orbits of the same period. In contrast to these periodic orbits there is a class of trajectories having unbounded path length and running along the real- $x$  axis. These are the only paths that violate time-reversal symmetry.

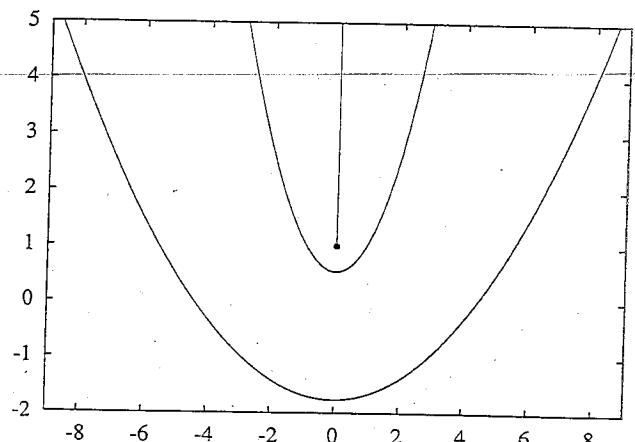


FIG. 10. Classical trajectories in the complex- $x$  plane for a particle described by the Hamiltonian  $H = p^2 - ix$  and having energy  $E = 1$ . Shown are parabolic trajectories and a turning point at  $i$ . All trajectories are unbounded.

$$H = p^2 + ix^3 \quad (\epsilon = 1)$$

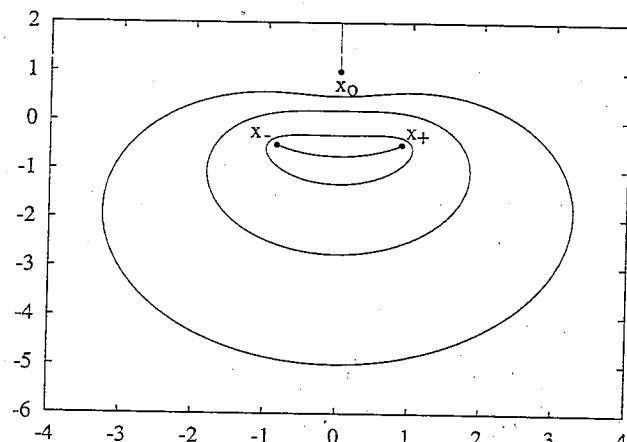


FIG. 2. Classical trajectories in the complex- $x$  plane for a particle described by the Hamiltonian  $H = p^2 + ix^3$  and having energy  $E = 1$ . An oscillatory trajectory connects the turning points  $x_{\pm}$ . This trajectory is enclosed by a set of closed, nested paths that fill the finite complex- $x$  plane except for points on the imaginary axis at or above the turning point  $x_0 = i$ . Trajectories originating at one of these exceptional points go off to  $i\infty$  or else they approach  $x_0$ , stop, turn around, and then move up the imaginary axis to  $i\infty$ .

$$H = p^2 + ix^7 \quad (\epsilon = 5)$$

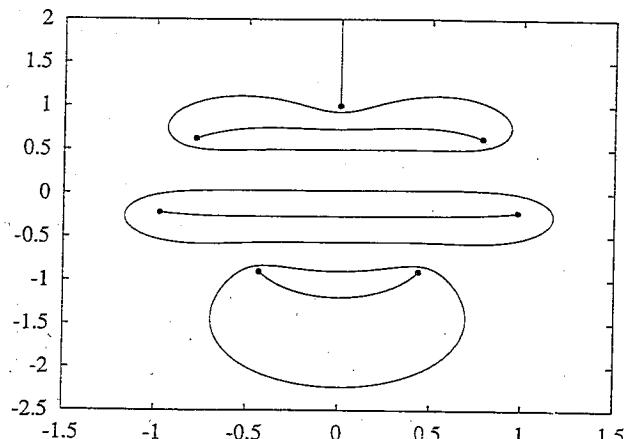


FIG. 6. Classical trajectories in the complex- $x$  plane for a particle described by the Hamiltonian  $H = p^2 + ix^7$  and having energy  $E = 1$ . Shown are oscillatory trajectories surrounded by periodic trajectories. Unbounded trajectories run along the positive-imaginary axis above  $x = i$ .

Derivation of an "energy" continuity-like equation:

$$i\hbar \partial_t \psi^{(+)}(z, t) = \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) \right] \psi^{(+)}(z, t) \quad | \cdot \psi^{(+)}(z, t)$$

$$-i\hbar \partial_t \psi^{(-)}(z, t) = \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) \right] \psi^{(-)}(z, t) \quad | \cdot \psi^{(-)}(z, t)$$

$$\Rightarrow \partial_t [\psi^{(-)}(z, t) \psi^{(+)}(z, t)] = -\frac{d}{dz} \left[ \frac{\hbar}{2m} \left( \psi^{(-)}(z, t) \frac{d}{dz} \psi^{(+)}(z, t) \right. \right. \\ \left. \left. - \psi^{(+)}(z, t) \frac{d}{dz} \psi^{(-)}(z, t) \right) \right]$$

$$\Rightarrow \underbrace{\partial_t [\psi^{(-)}(z, t) \psi^{(+)}(z, t)]}_{g(z, t)} = -\frac{d}{dz} \underbrace{\left[ \frac{\hbar}{2m} W(\psi^{(-)}(z, t), \psi^{(+)}(z, t)) \right]}_{j(z, t)}$$

Thresholds:

~~$j = S \cdot v \Rightarrow v = \frac{j}{S}$~~

$$\Rightarrow v = \frac{\frac{\hbar}{2m} W(\psi^{(-)}, \psi^{(+)})}{\psi^{(-)} \psi^{(+)}} = \frac{1}{2m} \left( (-i\hbar) \frac{\psi^{(+)}'}{\psi^{(+)}} - (-i\hbar) \frac{\psi^{(-)}'}{\psi^{(-)}} \right) \\ = \frac{1}{2} \left( \frac{p^{(+)}}{m} - \frac{p^{(-)}}{m} \right)$$

Spacial "probability" of the quantum particle along some "particle trajectory" for  $v \neq 0$ :

$$\frac{1}{T} \int_{t_0}^{t_1} dt = \frac{1}{T} \int_{z_0}^{z_1} dz \frac{1}{\dot{z}} = \frac{1}{T} \int_{z_0}^{z_1} dz \frac{1}{v} = \int_{z_0}^{z_1} dz \underbrace{\frac{2m}{\hbar T W[\psi^{(-)}, \psi^{(+)}]}}_{\text{"Metric"} \atop (\text{probabilistic weight})} \cdot \psi^{(-)} \psi^{(+)}$$

$\dot{z} \parallel v$

$$\partial_t [\psi^{(-)}(z, t) \psi^{(+)}(z, t)] = -\frac{d}{dz} \left[ \frac{\hbar}{2m_i} (\psi^{(-)}(z, t) \frac{d}{dz} \psi^{(+)}(z, t) - \psi^{(+)}(z, t) \frac{d}{dz} \psi^{(-)}(z, t)) \right]$$

Divide the equation by  $\psi^{(-)} \cdot \psi^{(+)}$ :

$$\Rightarrow \underbrace{\frac{\partial_t \psi^{(+)}}{\psi^{(+)}}} + \underbrace{\frac{\partial_t \psi^{(-)}}{\psi^{(-)}}} = \underbrace{-\frac{d}{dz} \left[ \frac{\hbar}{2m_i} \psi^{(-)} \psi^{(+)}' \right]}_{\psi^{(-)} \psi^{(+)}} + \underbrace{-\frac{d}{dz} \left[ -\frac{\hbar}{2m_i} \psi^{(+)} \psi^{(-)}' \right]}_{\psi^{(-)} \psi^{(+)}}$$

$$\underbrace{\frac{\partial_t \ln \psi^{(+)}}{\psi^{(+)}}} \quad \underbrace{\frac{\partial_t \ln \psi^{(-)}}{\psi^{(-)}}}_{\psi^{(-)}} \stackrel{?}{=} -\frac{d}{dz} j^{(+)} \quad \stackrel{?}{=} -\frac{d}{dz} j^{(-)}$$

Idea:

$$-\frac{d}{dz} \left[ \frac{\hbar}{2m_i} \psi^{(-)} \psi^{(+)}' \right] = -\frac{d}{dz} \left[ \frac{\hbar}{2m_i} \frac{\psi^{(-)} \psi^{(+)}'}{\psi^{(-)} \psi^{(+)}} \right] + \frac{\hbar}{2m_i} \psi^{(-)} \psi^{(+)}' \frac{d}{dz} \left( \frac{1}{\psi^{(-)} \psi^{(+)}} \right)$$

$$= -\frac{d}{dz} \left[ \frac{\hbar}{2m_i} \frac{\psi^{(+)}'}{\psi^{(+)}} \right] + \frac{\hbar}{2m_i} \psi^{(-)} \psi^{(+)}' \cdot (-1) \cdot \frac{\psi^{(-)}' (\psi^{(+)} + \psi^{(+)} \psi^{(-)})}{\psi^{(-)2} \psi^{(+)}2}$$

$$= -\frac{d}{dz} \left[ \frac{\hbar}{2m_i} \frac{\psi^{(+)}'}{\psi^{(+)}} \right] - \frac{\hbar}{2m_i} \underbrace{\frac{\psi^{(+)}'}{\psi^{(+)}} \cdot \left[ \frac{\psi^{(-)}'}{\psi^{(-)}} + \frac{\psi^{(+)} \psi^{(-)}}{\psi^{(+)}} \right]}_{\text{Divergence? Contribution of metric?}}$$

Leading term for  $j^{(+)}$

Idea:

$$\frac{1}{T} \int_{t_0}^{t_1} dt = \frac{1}{T} \int_{z_0}^{z_1} dz \frac{1}{V^{(+)}} = \int_{z_0}^{z_1} dz \underbrace{\frac{1}{T} \frac{\psi^{(+)}}{j^{(+)}}}_{\text{Probability density along "particle trajectory"}}$$

Probability density along "particle trajectory"

Recall: "Metric"  $\eta$  transforms theory into equivalent theory

$$H_\eta = \eta H \eta^{-1}, |\psi^{(\pm)}(t)\rangle_\eta = \eta |\psi^{(\pm)}(t)\rangle$$

$$\langle\langle \psi^{(\pm)}(t) | = \langle\langle \psi^{(\pm)}(t) | \eta^{-1}$$

$$\pm i\hbar \partial_t |\psi^{(\pm)}(t)\rangle_\eta = H_\eta |\psi^{(\pm)}(t)\rangle_\eta$$

$$\pm i\hbar \partial_t \langle\langle \psi^{(\pm)}(t) | = \langle\langle \psi^{(\pm)}(t) | H_\eta$$

$$((i\hbar\partial_t)^2 - H_\eta^2) |\psi(t)\rangle_\eta = 0$$

$$\langle\langle \psi(t) | ((i\hbar\overset{\leftarrow}{\partial}_t)^2 - H_\eta^2) = 0$$

$$|\psi^{(\pm)}(t)\rangle_\eta = e^{\pm \frac{1}{i\hbar} H_\eta t} |\psi^{(\pm)}(0)\rangle_\eta$$

$$\langle\langle \psi^{(\pm)}(t) | = \langle\langle \psi^{(\pm)}(0) | e^{\pm \frac{1}{i\hbar} H_\eta t}$$

$$|\psi(t)\rangle_\eta = |\psi^{(+)}(t)\rangle_\eta + |\psi^{(-)}(t)\rangle_\eta$$

$$\langle\langle \psi(t) | = \langle\langle \psi^{(+)}(t) | + \langle\langle \psi^{(-)}(t) |$$

## Non-Hermitian Quantum Theory

⇒ retarded/advanced Schrödinger equations

$$\pm i\hbar \partial_t |\psi^{(\pm)}(t)\rangle = H |\psi^{(\pm)}(t)\rangle$$

$$\pm i\hbar \partial_t \langle\langle \psi^{(\pm)}(t) | = \langle\langle \psi^{(\pm)}(t) | H$$

or equivalently:

$$\pm i\hbar \partial_t |\psi^{(\pm)}(t)\rangle = H |\psi^{(\pm)}(t)\rangle$$

$$\pm i\hbar \partial_t \gamma \langle\langle \psi^{(\pm)}(t) | = \gamma \langle\langle \psi^{(\pm)}(t) | H \gamma$$

Particular case (e.g. PT-symmetry):  $\boxed{H\gamma = H^+}$

$$\Rightarrow \pm i\hbar \partial_t |\psi^{(\pm)}(t)\rangle = H |\psi^{(\pm)}(t)\rangle$$

$$\pm i\hbar \partial_t \underbrace{\langle\langle \psi^{(\pm)}(t) |}_{\langle\psi^{(\mp)}(t)|} = \underbrace{\gamma \langle\langle \psi^{(\pm)}(t) |}_{\langle\psi^{(\mp)}(t)|} H^+$$

$$\text{Hence: } H\gamma = \gamma H\gamma^{-1} = H^+ \Rightarrow \langle\langle \psi^{(\mp)}(t) | = \underbrace{\gamma \langle\langle \psi^{(\pm)}(t) |}_{\langle\langle \psi^{(\pm)}(t) |} \gamma^{-1}$$

# Quantization of $2 \times 2$ matrix-Hamiltonian with definite metric

$$\mathcal{H} = E_1 a_1^+ a_1 + E_2 a_2^+ a_2 = (a_1^+, a_2^+) \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= \underbrace{(a_1^+, a_2^+) X^{-1}}_{(\bar{a}_1^+, \bar{a}_2^+)} \times \underbrace{\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}}_H \times^{-1} \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix}}$$

$$= \underbrace{(a_1^+, a_2^+) X^+}_{(a_1'^+, a_2'^+)} \underbrace{X^{+1} X^{-1}}_{\eta} H \underbrace{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}_{\begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix}}$$

Hence:  $\mathcal{H} = (\bar{a}_1^+, \bar{a}_2^+) H \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = (a_1'^+, a_2'^+) \underbrace{(X X^+)^{-1}}_{\eta} H \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix}$

with:  $\begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} = X \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, (\bar{a}_1^+, \bar{a}_2^+) = (a_1^+, a_2^+) X^{-1}$

Commutation relations:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (a_1^+, a_2^+) - \begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} (a_1, a_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} (\bar{a}_1^+, \bar{a}_2^+) - X X^T \begin{pmatrix} \bar{a}_1^+ \\ \bar{a}_2^+ \end{pmatrix} (a_1'^+, a_2'^+) (X X^T)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} (a_1'^+, a_2'^+) - X X^{*-1} \begin{pmatrix} a_1'^+ \\ a_2'^+ \end{pmatrix} (a_1'^+, a_2'^+) (X X^{*-1})^T = X X^+$$

Unfortunately an (anti)holomorphic space  
contains some indefinite metric:

Start with  $x_1, x_2 \in \mathbb{R}$  and  $\frac{d}{dx_1}, \frac{d}{dx_2}$ .

$$\Rightarrow \begin{pmatrix} \left[ \frac{d}{dx_1}, x_1 \right] & \left[ \frac{d}{dx_1}, x_2 \right] \\ \left[ \frac{d}{dx_2}, x_1 \right] & \left[ \frac{d}{dx_2}, x_2 \right] \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Standard harmonic oscillator:

$$x_1 = \frac{i}{\sqrt{2}}(a_1 - a_1^+) \quad x_2 = \frac{i}{\sqrt{2}}(a_2 - a_2^+)$$

$$\frac{d}{dx_1} = \frac{i}{\sqrt{2}}(a_1 + a_1^+) \quad \frac{d}{dx_2} = \frac{i}{\sqrt{2}}(a_2 + a_2^+)$$

$$\begin{pmatrix} [a_1, a_1^+] [a_1, a_1^+] \\ [a_2, a_2^+] [a_2, a_2^+] \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Change to (anti)holomorphic coordinates:

$$\xi = \frac{x_1 + ix_2}{\sqrt{2}}$$

$$\xi^* = \frac{x_1 - ix_2}{\sqrt{2}}$$

$$\begin{pmatrix} \left[ \frac{d}{d\xi}, \xi \right] \left[ \frac{d}{d\xi}, \xi^* \right] \\ \left[ \frac{d}{d\xi^*}, \xi \right] \left[ \frac{d}{d\xi^*}, \xi^* \right] \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{d}{d\xi} = \frac{1}{\sqrt{2}} \left( \frac{d}{dx_1} + \frac{1}{i} \frac{d}{dx_2} \right)$$

$$\frac{d}{d\xi^*} = \frac{1}{\sqrt{2}} \left( \frac{d}{dx_1} - \frac{1}{i} \frac{d}{dx_2} \right)$$

$$\Rightarrow \xi = \frac{i}{2}(a_1 + ia_2 - (a_1^+ + ia_2^+)) = \frac{i}{\sqrt{2}} \left( \underbrace{\frac{a_1 - ia_2^+}{\sqrt{2}}}_{a} - \underbrace{\frac{a_1^+ - ia_2}{\sqrt{2}}}_{a^+} \right) = \frac{i}{\sqrt{2}}(a - a^+)$$

$$\xi^* = \frac{i}{2}(a_1 - ia_2 - (a_1^+ - ia_2^+)) = \frac{i}{\sqrt{2}} \left( \underbrace{\frac{a_1 + ia_2^+}{\sqrt{2}}}_{c} - \underbrace{\frac{a_1^+ + ia_2}{\sqrt{2}}}_{a^+} \right) = \frac{i}{\sqrt{2}}(c - a^+)$$

Convince yourself that there holds the indefinite metric:

$$\begin{pmatrix} [a, a^+] [a, c^+] \\ [c, a^+] [c, c^+] \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{and } \begin{cases} \frac{d}{d\xi} = \frac{i}{\sqrt{2}}(a + a^+) \\ \frac{d}{d\xi^*} = \frac{i}{\sqrt{2}}(c + a^+) \end{cases}$$

# Quantization of $2 \times 2$ matrix-Hamiltonian with indefinite metric

$$\mathcal{H} = E_1 c^+ a + E_2 a^+ c = (c^+, a^+) \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}$$

$$= \underbrace{(c^+, a^+) X^{-1}}_{(\bar{c}^T, \bar{a}^T)} \underbrace{\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}}_H \underbrace{X^{-1} \begin{pmatrix} a \\ c \end{pmatrix}}_{\begin{pmatrix} a' \\ c' \end{pmatrix}}$$

$$= \underbrace{(a^+, c^+) X^+}_{(a'^+, c'^+)} \underbrace{(X \sigma_1 X^+)^{-1}}_\eta H \underbrace{X \begin{pmatrix} a \\ c \end{pmatrix}}_{\begin{pmatrix} a' \\ c' \end{pmatrix}}$$

Hence:  $\mathcal{H} = (\bar{c}^T, \bar{a}^T) H \begin{pmatrix} a' \\ c' \end{pmatrix} =$

$$= \underbrace{(\bar{a}^T, \bar{c}^T) \sigma_1}_{} H \underbrace{\begin{pmatrix} a' \\ c' \end{pmatrix}}_{\begin{pmatrix} a' \\ c' \end{pmatrix}} = \underbrace{(a'^+, c'^+)}_{(a^+, c^+) X^{-1}} \underbrace{(X \sigma_1 X^+)^{-1} H}_{\eta} \underbrace{\begin{pmatrix} a' \\ c' \end{pmatrix}}_{X \begin{pmatrix} a \\ c \end{pmatrix}}$$

Commutation relations:

$$\begin{pmatrix} a \\ c \end{pmatrix} (a^+, c^+) - \begin{pmatrix} a^+ \\ c^+ \end{pmatrix} (a, c) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

$$\Rightarrow \begin{pmatrix} a' \\ c' \end{pmatrix} (\bar{c}^T, \bar{a}^T) - X \sigma_1 X^T \begin{pmatrix} \bar{c}^T \\ \bar{a}^T \end{pmatrix} (a'^+ c') (X \sigma_1 X^T)^{-1} = \mathbb{1}$$

$$\Rightarrow \begin{pmatrix} a' \\ c' \end{pmatrix} (\bar{a}^T, \bar{c}^T) - X \sigma_1 X^T \sigma_1 \begin{pmatrix} \bar{a}^T \\ \bar{c}^T \end{pmatrix} (a', c') (X \sigma_1 X^T)^{-1} = \sigma_1$$

$$\Rightarrow \begin{pmatrix} a' \\ c' \end{pmatrix} (a'^+, c'^+) - X X^{*-1} \begin{pmatrix} a'^+ \\ c'^+ \end{pmatrix} (a', c') (X X^{*-1})^+ = X \sigma_1 X^+$$

# Non-Hermitian Quantum-Field Theory and Quantum Mechanics → sketching the relation:

Start with:  $\mathcal{H} = E c^\dagger \alpha + E^* \alpha^\dagger c$  (diagonal!)

$$\Rightarrow \psi(t) = e^{\frac{1}{i\hbar} H t} \psi(0) e^{-\frac{1}{i\hbar} H t}$$

$$= e^{\frac{1}{i\hbar} H t} (c^\dagger + \alpha) e^{-\frac{1}{i\hbar} H t}$$

$$= \underbrace{e^{\frac{1}{i\hbar} H t} c^\dagger e^{-\frac{1}{i\hbar} H t}}_{\psi^{(+)}(t)} + \underbrace{e^{+\frac{1}{i\hbar} H t} \alpha e^{-\frac{1}{i\hbar} H t}}_{\psi^{(-)}(t)}$$

$$\Rightarrow \psi(t) = e^{\frac{1}{i\hbar} E t} c^\dagger + e^{-\frac{1}{i\hbar} E t} \alpha$$

$$\psi^+(t) = e^{\frac{1}{i\hbar} E^* t} \alpha^\dagger + e^{-\frac{1}{i\hbar} E^* t} c$$

$$|\psi^{(+)}(t)\rangle = \psi(t)|0\rangle = e^{\frac{1}{i\hbar} E t} c^\dagger |0\rangle$$

$$\langle\langle\psi^{(-)}(t)\rangle\rangle = \langle 0|\psi(t) = \langle 0| \alpha e^{-\frac{1}{i\hbar} E t}$$

$$|\psi^{(-)}(t)\rangle = \psi^+(t)|0\rangle = e^{+\frac{1}{i\hbar} E^* t} \alpha^\dagger |0\rangle$$

$$\langle\psi^{(+)}(t)\rangle = \langle 0|\psi^+(t) = \langle 0| c e^{-\frac{1}{i\hbar} E^* t}$$

Compare to Quantum-Field Theory ( $\rightarrow$  Kleefeld):  
 $(KG\text{-theory} \Leftrightarrow \text{Harmon. Osz.})$

$$\Psi(z) = \int \frac{d^3 p}{(2\pi)^3 2\omega(\vec{p})} [e^{-ipz} c^\dagger(\vec{p}) + e^{ipz} \alpha(\vec{p})] \Big|_{p^0 = \omega(\vec{p})}$$

$$\Psi^+(z) = \int \frac{d^3 p}{(2\pi)^3 2\omega^*(\vec{p})} [e^{-ip^* z} \alpha^\dagger(\vec{p}) + e^{ip^* z} c(\vec{p})] \Big|_{p^{0*} = \omega^*(\vec{p})}$$

# How about non-Hermitian Quantum Theory and the Correspondence Principle?

by

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## KG-field (Neutral):

Comment to I. Antoniou

$$\text{Naively } \mathcal{L} = \frac{1}{2}((\partial\varphi)^2 - m^2\varphi^2) \quad m \in \mathbb{R}, \varphi = \varphi^+$$

$$[\alpha, \alpha^\dagger] = 1$$

$$[\alpha, \alpha] = 0$$

$$[\alpha^\dagger, \alpha^\dagger] = 0$$

Problem: Causal propagation

$$\frac{1}{p^2 - m^2 + i\epsilon}$$

Correctly: Causal KG-field / resonance

(Remember  $g^{tt*} = (+, -, -, -)$  indefinite)

$$\mathcal{L} = \underbrace{\frac{1}{2}((\partial\varphi)^2 - M^2\varphi^2)}_{\substack{\text{causal} \\ \text{"Gamov"}}} + \underbrace{\frac{1}{2}((\partial\varphi^*)^2 - M^{*2}\varphi^{*2})}_{\substack{\text{anticausal} \\ \text{"Anti-Gamov"}}}$$

$$\varphi \neq \varphi^* \quad \text{as } M^2 \neq M^{*2}$$

$$\begin{pmatrix} [\alpha, \alpha^\dagger] & [\alpha, \alpha^\dagger] \\ [\alpha^\dagger, \alpha^\dagger] & [\alpha^\dagger, \alpha^\dagger] \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \text{"indefinite metric"}$$

$$\text{Recall: } \varphi = \int \frac{d^3 p}{(2\pi)^3 2\omega} [\alpha(\vec{p}) e^{-ipx} + c(\vec{p}) e^{+ipx}] \Big|_{p^0 = \omega}$$

$$\varphi^* = \int \frac{d^3 p}{(2\pi)^3 2\omega^*} [c(\vec{p}) e^{-ipx} + \alpha^*(\vec{p}) e^{+ipx}] \Big|_{p^0 = \omega^*}$$

$$\text{As } \varphi \neq \varphi^* \Rightarrow \varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2), \varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2)$$

$$\varphi_1 = \varphi_1^*, \varphi_2 = \varphi_2^*$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}((\partial\varphi)^2 - M^2\varphi^2) + \frac{1}{2}((\partial\varphi^*)^2 - M^{*2}\varphi^{*2}) =$$

$$= \frac{1}{2}((\partial\varphi_1)^2 - \text{Re}[M^2]\varphi_1^2) - \frac{1}{2}((\partial\varphi_2)^2 - \text{Re}[M^2]\varphi_2^2)$$

$$+ \text{const} \times \text{Im}[M^2] \varphi_1 \varphi_2 \leftarrow \begin{array}{l} \text{Interaction introduced} \\ \text{"Mixing"} \end{array}$$

by causality & Lorentz-inv.

⇒ Theory with Hermitian asymptotic states and resonances ill →

as non-analytic interaction terms &  $\varphi\varphi^* \Rightarrow$  violation of causality, locality, analyticity