Solvable $\mathcal{PT}$-symmetric potentials beyond the shape-invariant class

Géza Lévai

Institute of Nuclear Research of the Hungarian Academy of Sciences (ATOMKI), Debrecen, Hungary

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1. Exactly solvable potentials in general

A variable transformation method

Classification of solvable potentials:

shape-invariant (up to 3 par.) $\in$ Natanzon class (6 par.)

2. Exactly solvable $\mathcal{PT}$-symmetric potentials

Adaptating the techniques to $\mathcal{PT}$: $V_R(-x) = V_R(x)$ $V_I(-x) = -V_I(x)$

Exact results for shape-invariant potentials: $\Psi_n(x)$, E_n, $T(k)$, $R(k)$, $\mathcal{C}$

BUT: Very few studies on $\mathcal{PT}$-symmetric potentials beyond the SI class

3. A new exactly solvable potential class:

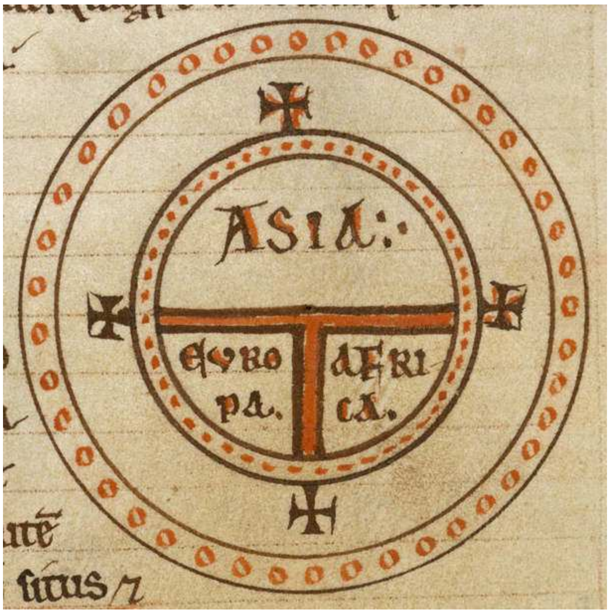
A 4 parameter subset of the Natanzon class

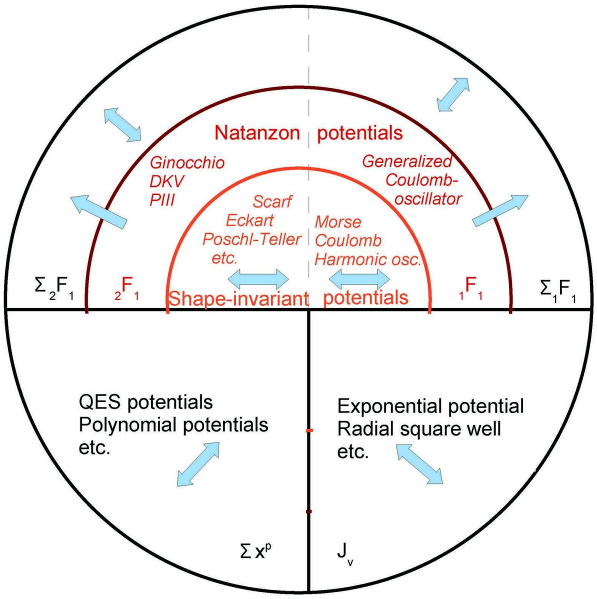
Continuous transformation between pairs of shape-invariant limits

Bound-state energies defined implicitly by a quartic equation

Different properties in four different parameter domains

1. Exactly solvable potentials in general





Let's focus on the top left quarter with ${}_2F_1$

i.e. Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ for bound states

Exact solutions of the Schrödinger equation

An old method *Bhattacharjie and Sudarshan 1962*

Variable transformation:

Schrödinger eq. $\Longrightarrow$ differential equation of special function F now $P_n^{(\alpha,\beta)}(z)$

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + (E - V(x))\psi(x) = 0 \qquad \text{insert} \qquad \psi(x) = \mathbf{f}(x) P_n^{(\alpha,\beta)}(\mathbf{z}(x))$$

and compare with the Jacobi differential equation

to get

$$\begin{aligned} E - V(x) &= \frac{z'''(x)}{2z'(x)} - \frac{3}{4}\left(\frac{z''(x)}{z'(x)}\right)^2 + \frac{(z'(x))^2}{1 - z^2(x)}\left(n + \frac{\alpha+\beta}{2}\right)\left(n + \frac{\alpha+\beta}{2} + 1\right) \\ &\quad + \frac{(z'(x))^2}{(1 - z^2(x))^2}\left[1 - \left(\frac{\alpha+\beta}{2}\right)^2 - \left(\frac{\alpha-\beta}{2}\right)^2\right] \\ &\quad - \frac{2z(x)(z'(x))^2}{(1 - z^2(x))^2}\left(\frac{\alpha+\beta}{2}\right)\left(\frac{\alpha-\beta}{2}\right). \end{aligned}$$

The solutions are

$$\psi(x) \sim (z'(x))^{-\frac{1}{2}}(1+z(x))^{\frac{\beta+1}{2}}(1-z(x))^{\frac{\alpha+1}{2}}P_n^{(\alpha,\beta)}(z(x)) \ .$$

The yet unknown $\mathbf{z}(x)$ can be obtained from a differential equation

$$\boxed{\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\phi(z) \equiv \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\frac{p_{\mathrm{I}}(1-z^2)+p_{\mathrm{II}}+p_{\mathrm{III}}z}{(1-z^2)^2} = C \ .}$$

by direct integration

$$\int \phi^{1/2}(\mathbf{z})\mathrm{d}\mathbf{z} = C^{1/2}x + \epsilon \ .$$

ϵ: integration constant, coordinate shift

This is to generate a **constant term** on the r.h.s. of $E - V(x) = ...$

Note: sometimes only $x(\mathbf{z})$ can be determined $\Longrightarrow$ implicit potentials

The list of shape-invariant potentials solved by Jacobi polynomials

$(z')^2 =$ (Class)	C	$z(x)$	$V(x)$	$x \in$	Name
$C(1-z^2)$ (PI)	$-a^2$	$\mathrm{i}\sinh(x)$	$(B^2 - A^2 - A)\mathrm{sech}^2(ax)$ $+B(2A+1)\mathrm{sech}(ax)\tanh(ax)$	$(-\infty, \infty)$	Scarf II
	$-a^2$	$\cosh(ax)$	$(B^2 + A^2 + A)\mathrm{cosech}^2(ax)$ $-B(2A+1)\mathrm{cosech}(ax)\coth(ax)$	$[0, \infty)$	gen. Pöschl–Teller
	a^2	$\sin(ax)$	$(B^2 + A^2 - A)\sec^2(ax)$ $-B(2A-1)\sec(ax)\tan(ax)$	$[-\frac{\pi}{2a}, \frac{\pi}{2a}]$	Scarf I
	$4a^2$	$\cos(2ax)$	$A(A-1)\sec^2(ax)$ $+B(B-1)\mathrm{cosec}^2(ax)$	$[0, \frac{\pi}{2a}]$	Pöschl–Teller I
	$-4a^2$	$\cosh(2ax)$	$-A(A+1)\mathrm{sech}^2(ax)$ $+B(B-1)\mathrm{cosech}^2(ax)$	$[0, \infty)$	Pöschl–Teller II
$C(1-z^2)^2$ (PII)	a^2	$\tanh(ax)$	$-A(A+1)\mathrm{sech}^2(ax)$ $+2B\tanh(ax)$	$(-\infty, \infty)$	Rosen–Morse II
	a^2	$\coth(ax)$	$A(A-1)\mathrm{cosech}^2(ax)$ $-2B\coth(ax)$	$[0, \infty)$	Eckart
	$-a^2$	$\mathrm{i}\tan(ax)$	$A(A+1)\sec^2(ax)$ $+2B\tan(ax)$	$[-\frac{\pi}{2a}, \frac{\pi}{2a}]$	Rosen–Morse I

Obtained by selecting certain single terms on the right handside of $E - V(x) = ...$

The Pöschl–Teller I and II potentials are equivalent with the Scarf I and II

How to $\mathcal{PT}$-symmetrize these?

2. Exactly solvable $\mathcal{PT}$-symmetric potentials

First a brief historical overview

Numerical results came first *Bender and Boettscher 1998*

Results for $V(x) = x^2(\mathrm{i}x)^\delta$

Real eigenvalues for $\delta > 0$

Complex eigenvalues appear gradually for $\delta < 0$

Trajectory off the real line for $\delta > 2$

Semi-analytical example *Znojil and Lévai 2001*

The $\mathcal{PT}$ symmetric square well

Gradual mechanism of the spontaneous breakdown of $\mathcal{PT}$ symmetry

What about exactly solvable examples?

How to $\mathcal{PT}$-symmetrize exactly solvable potentials

G. Lévai, M. Znojil, J. Phys A **33** (2000) 7165.

Adjust potential parameters — making use of $\mathcal{PT} z(x) = \pm z(x)$

Directly applicable to potentials defined on $x \in (-\infty, \infty)$ — Scarf II, Rosen–Morse II

Reconsider boundary conditions — avoid singularities

imaginary integration constant $\epsilon = -\mathrm{i}c$

Apply it to potentials defined on $x \in (0, \infty)$,

$z(x)$	$x \in$	$y = x - \mathrm{i}c$	$\mathcal{PT} z(y) = \pm z(y)$	Example
$\sinh(x)$	$(-\infty, \infty)$		$-\sinh(y)$	Scarf II
$\cosh(x)$	$(-\infty, \infty)$		$\cosh(y)$	gen. Pöschl–Teller
$\sin(x)$	$(-\pi/2, \pi/2)$		$-\sin(z)$	Scarf I

Note: Introducing $\mathrm{i}c$ does not change the spectrum

...so it cannot introduce spontaneous $\mathcal{PT}$ breaking

In some other cases the problem has to be defined on a more general trajectory

The boundary conditions cannot be satisfied on $y = x - \mathrm{i}c$

This is the case for the Coulomb and Morse potentials with $F(z) = L_n^{(\alpha)}(z)$

The guinea pig: The Scarf II potential

$$V(x) = -\frac{1}{\cosh^2 x}\left[\left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha-\beta}{2}\right)^2 - \frac{1}{4}\right] + \frac{2\mathrm{i}\sinh x}{\cosh^2 x}\left(\frac{\beta+\alpha}{2}\right)\left(\frac{\beta-\alpha}{2}\right)$$

Relations for the parameters:

$\mathcal{PT}$ symmetry:	$\Longrightarrow$	α, β are real **or** imaginary
$\alpha \leftrightarrow \beta$:	$\Longrightarrow$	$V(x) \leftrightarrow V(-x)$
$V(x)$ invariant under $\alpha \leftrightarrow -\alpha$	$\Longrightarrow$	$q\alpha \equiv \pm\alpha$ quasi-parity

$$\psi_n^{(q)}(x) = C_n^{(q)}(1-\mathrm{i}\sinh(x+\mathrm{i}\epsilon))^{\frac{q\alpha}{2}+\frac{1}{4}}(1+\mathrm{i}\sinh(x+\mathrm{i}\epsilon))^{\frac{\beta}{2}+\frac{1}{4}}P_n^{(q\alpha,\beta)}(\mathrm{i}\sinh(x+\mathrm{i}\epsilon))$$

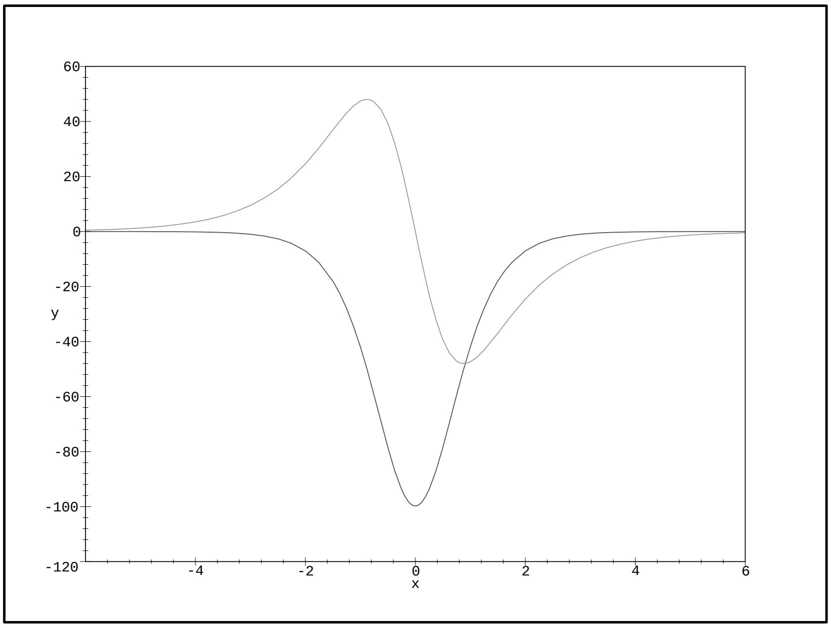
Normalizable if $\quad n^{(q)} < -[\mathrm{Re}(q\alpha+\beta)+1]/2$

The second set corresponds to resonances in the Hermitian setting ($\alpha^* = \beta$)

$$E_n^{(q)} = -\left(n + \frac{q\alpha+\beta+1}{2}\right)^2$$

Complex conjugate pairs if α is imaginary   Spontaneous breakdown of $\mathcal{PT}$ symmetry

"Sudden" mechanism: all the $E_n^{(q)}$ turn complex at the same time



$$\alpha = -2, \quad \beta = -14 \qquad\qquad n^{(+)} < 8, \qquad n^{(-)} < 6, \qquad E_n^{(q)} < 0 \quad \text{for } \mathbf{all}\ n$$

The black sheep: The Rosen–Morse II potential

G. Lévai, E. Magyari, J. Phys. A **42** (2009) 195302

$$V(x) = -\frac{s(s+1)}{\cosh^2(x)} + 2\mathrm{i}\lambda \tanh(x)$$

$\mathcal{PT}$ symmetry: $\Longrightarrow$ $s(s+1)$, λ are real

Bound-state solutions:

$$\psi_n(x) = C_n(1-\tanh x)^{\frac{\alpha}{2}}(1+\tanh x)^{\frac{\beta}{2}} P_n^{(\alpha,\beta)}(\tanh x)$$

$$\alpha_n = s - n + \frac{\mathrm{i}\lambda}{s-n}\ , \qquad \beta_n = s - n - \frac{\mathrm{i}\lambda}{s-n}$$

Only one solution can be regular at the same time $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$

$$E_n = -(s-n)^2 + \frac{\lambda^2}{(s-n)^2}\ , \qquad n = 0,\ 1,\ \ldots n_{\max} < s\ .$$

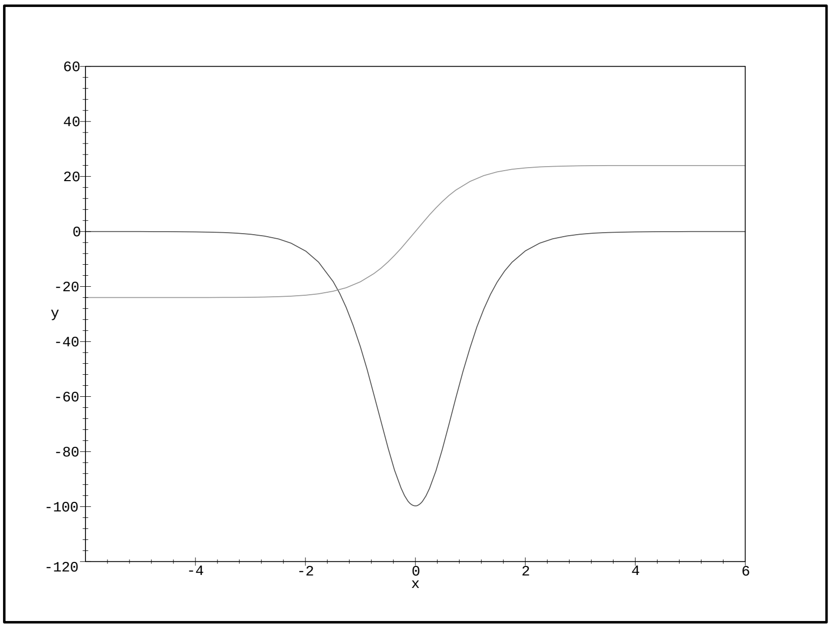
Peculiarities of the spectrum

No quasi-parity

No complex energy eigenvalues **no** spontaneous breakdown of $\mathcal{PT}$ symmetry

But $E_n > 0$ if $s - |\lambda|^{1/2} \leq n < s$ increasing non-hermiticity

Furthermore for $|\lambda| > s^2$ **all** $E_n > 0$



$s = 9.5, \quad \lambda = 12$ $\qquad n \leq 9,$ $\qquad V_R(x)$ is the same as for the Scarf II potential

Is the difference due to the dominant imaginary component?

$$V_R(\pm\infty) = 0 \qquad \text{BUT} \qquad V_I(\pm\infty) = \pm 2\mathrm{i}\lambda \neq 0$$

An old friend: The DKV potential

outside the SI class

R. Dutt, A. Khare, Y. P. Varshni, J. Phys. A **28** (1995) L107

$$\begin{aligned} V(x) &= -B[1+\exp(-2x)]^{-1/2} + A[1+\exp(-2x)]^{-1} - \frac{3}{4}[1+\exp(-2x)]^{-2} \\ &= -2B[1+\tanh(x)]^{1/2} - \left(\frac{A}{4} - \frac{3}{32}\right)\tanh(x) + \frac{3}{64}\cosh^{-2}(x) \end{aligned}$$

$$\psi_n(x) = C_n z^{1/2}(x)(z(x)+1)^{\beta_n/2}(z(x)-1)^{\alpha_n/2} P_n^{(\alpha_n,\beta_n)}(z(x)) \qquad z(x) = [1+\mathrm{e}^{-2x}]^{1/2}$$

$$E_n = -\left(n + \frac{\alpha_n+\beta_n+1}{2}\right)^2$$

$\alpha_n + \beta_n$ is determined implicitly by a cubic algebraic equation

Normalizability requires $n + \frac{\alpha_n+\beta_n+1}{2} < 0$

Always the lowest real root yields E_n

Identified as a Natanzon-class potential

R. Roychoudhury, P. Roy, M. Znojil, G. Lévai, J. Math. Phys. **42** (2001) 1996

How to $\mathcal{PT}$-symmetrize the DKV potential?

Not separable to even and odd components...

BUT take $x \to \mathrm{i}x$

M. Znojil, G. Lévai, P. Roy, R. Roychoudhury, Phys. Lett. A **290** (2001) 249

Formally everything remains the same

hyperbolic functions $\longrightarrow$ trigonometric functions PLUS i factors

Now the acceptable energy eigenvalues will be complex conjugate pairs

obtained by two complex conjugate roots of the cubic equation

Spontaneous breakdown of $\mathcal{PT}$ symmetry

But are there real igenvalues too?

What do we know about solvable $\mathcal{PT}$-symmetric potentials?

	defined on	Sp. $\mathcal{PT}$ br.	q	Ps. norm	$T(k)$, $R(k)$	etc.
Scarf II	x	sudden	Y	$\neq(-1)^n$	Y	
gen. Pöschl–Teller	$x-\mathrm{i}c$	sudden	Y		Y	
Scarf I	confined	No	No	$(-1)^n$	n.a.	$\mathcal{C}$
Rosen–Morse II	x	No	No	$(-1)^n$	Y	
Eckart	$x-\mathrm{i}c$	sudden	Y			
Rosen–Morse I	confined	No	No	$(-1)^n$	n.a.	
DKV	x	?	?			

3. A new exactly solvable potential class

Remember the equation defining the $z(x)$ function:

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \phi(z) \equiv \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 \frac{p_{\mathrm{I}}(1-z^2)+p_{\mathrm{II}}+p_{\mathrm{III}}z}{(1-z^2)^2} = C\ .$$

Shape-invariant potentials obtained for $p_{\mathrm{I}} \neq 0$ or $p_{\mathrm{II}} \neq 0$

Now take a combination: $p_{\mathrm{I}} = 1, \quad p_{\mathrm{II}} = \delta$

Implicit potential: only $x(z)$ is known in closed form...

...nevertheless, everything can be evaluated **exactly**

The actual form of the potential:

$$E - V(x) = \left(n + \frac{\alpha+\beta+1}{2}\right)^2 - \frac{3C\delta}{4}\frac{(3\delta+2)}{(\delta+1-z^2(x))^2} + \frac{5C\delta^2}{4}\frac{(\delta+1)}{(\delta+1-z^2(x))^3} - \frac{C\Sigma}{\delta+1-z^2(x)} - \frac{2C\Lambda z(x)}{\delta+1-z^2(x)}$$

$$\Sigma = \delta\left(n + \frac{\alpha+\beta+1}{2}\right)^2 - \delta + \left(\frac{\alpha+\beta}{2}\right)^2 + \left(\frac{\alpha-\beta}{2}\right)^2 - \frac{1}{4}$$

$$\Lambda = \frac{\alpha+\beta}{2}\frac{\alpha-\beta}{2} \qquad \alpha = \alpha_n\ , \qquad \beta = \beta_n$$

$$\Psi_n(x) = N_n(\delta+1-z^2(x))^{1/4}(1-z(x))^{\alpha/2}(1+z(x))^{\beta/2}P_n^{(\alpha,\beta)}(z(x))$$

A four-parameter (2+2) potential:

C and δ control the variable transformation

Σ and Λ set the coupling coefficients $\Lambda = 0$: symmetric Ginocchio case

A special case of the 3+3 parameter Natanzon class

What is always the same:

The potential, the energy expression, the wavefunction, the spectral equation:

$$(\delta+1)\omega^4 + \delta(2n+1)\omega^3 + \left(\frac{\delta}{4}(2n+1)^2 - \delta - \Sigma - \frac{1}{4}\right)\omega^2 + \Lambda^2 = 0$$

The defining differential equation

$$\frac{(z')^2}{(1-z^2)^2}(\delta+1-z^2) = C$$

What changes is:

The $x(z)$, i.e. the $z(x)$ function: it depends on C and δ

$C<0,\ \delta\geq 0 \implies z^2 \leq 0$ $z(x)$ imaginary and unbound

$C>0,\ \delta\geq 0 \implies 0 \leq z^2 \leq 1$ $z(x)$ real and bounded

$C<0,\ \delta\leq 0 \implies z^2 \geq 1$ $z(x)$ real and unbound

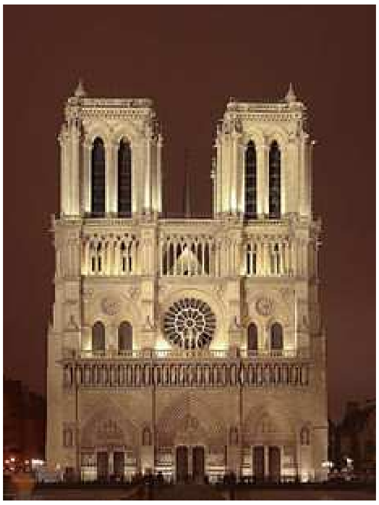
$z(x)$ combined with the boundary conditions restricts the ω roots and thus $E_n(\omega)$

$\mathcal{PT}\, z(x) = z^*(-x) = \pm z(x)$ also selects if Λ is real or imaginary

Note: for $C>0$, $\delta \leq 0$ $z(x)$ is complex and has no definite $\mathcal{PT}$ parity

What about the spontaneous breakdown of $\mathcal{PT}$ symmetry?

Paris is full of (slightly) broken symmetries

Not to mention the character of the left and the right bank...

The $C = -a^2 < 0$, $\delta \geq 0$ case: $z(x)$ imaginary and unbound, $z^2 \leq 0$, Λ real

The $x(z)$ solution:

vanishing Re z (*not to be confused with* Řež)

$$C^{1/2}x = \arctan[z(\delta+1-z^2)^{-1/2}] + \delta^{1/2}\mathrm{Artanh}[\delta^{1/2}z(\delta+1-z^2)^{-1/2}] \ .$$

$\delta \to 0$: $z(x) = \mathrm{i}\sinh(ax)$ Scarf II limit complex E_n exist

$\delta \to \infty$: $z(x) = \mathrm{i}\tan(ax)$ Rosen–Morse I limit no complex E_n exist

The wavefunctions:

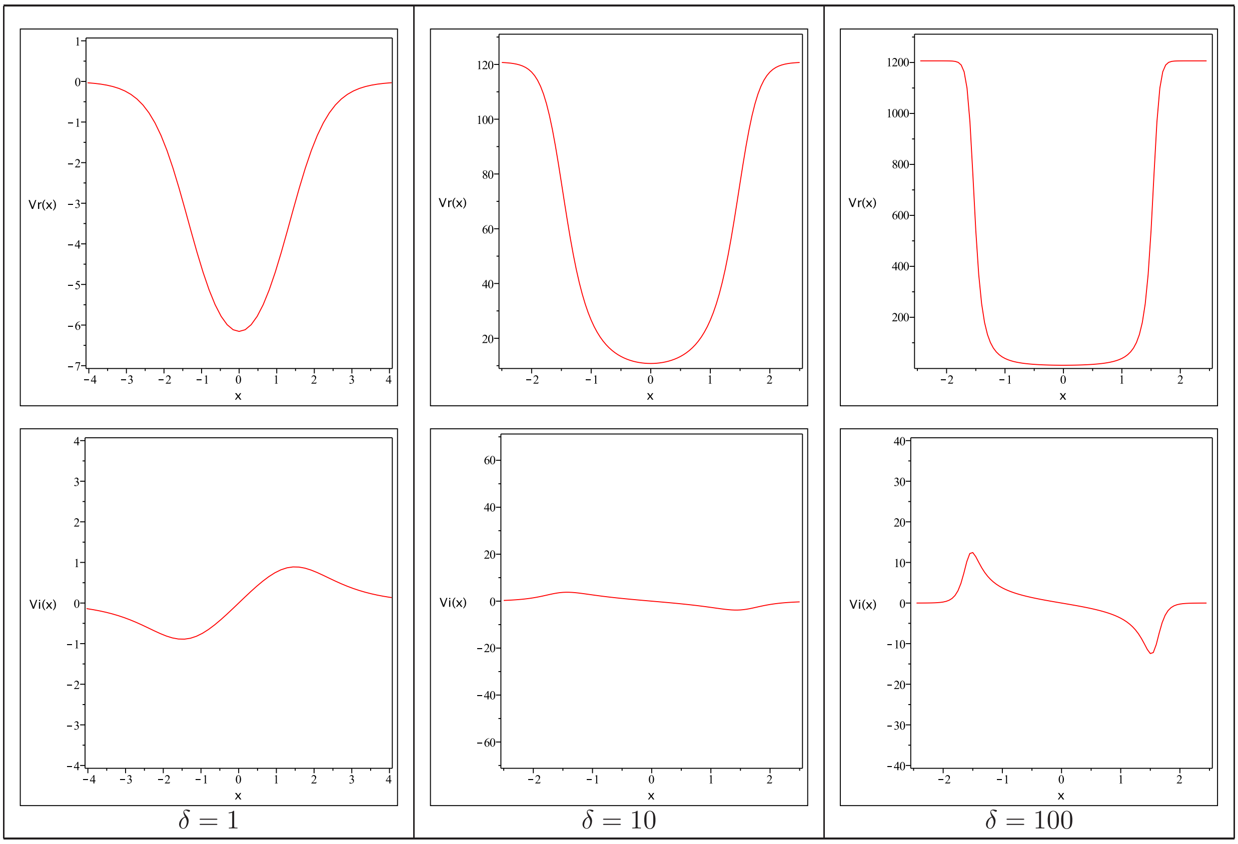
$$\Psi_n(x) = N_n(\delta+1-z^2(x))^{1/4}(1-z(x))^{\alpha/2}(1+z(x))^{\beta/2}P_n^{(\alpha,\beta)}(\mathrm{i}\sinh(x))$$

$$\alpha = \omega + \tfrac{\Lambda}{\omega}, \qquad \beta = \omega - \tfrac{\Lambda}{\omega}$$

Normalizability condition: Re $\omega < -n - \frac{1}{2}$

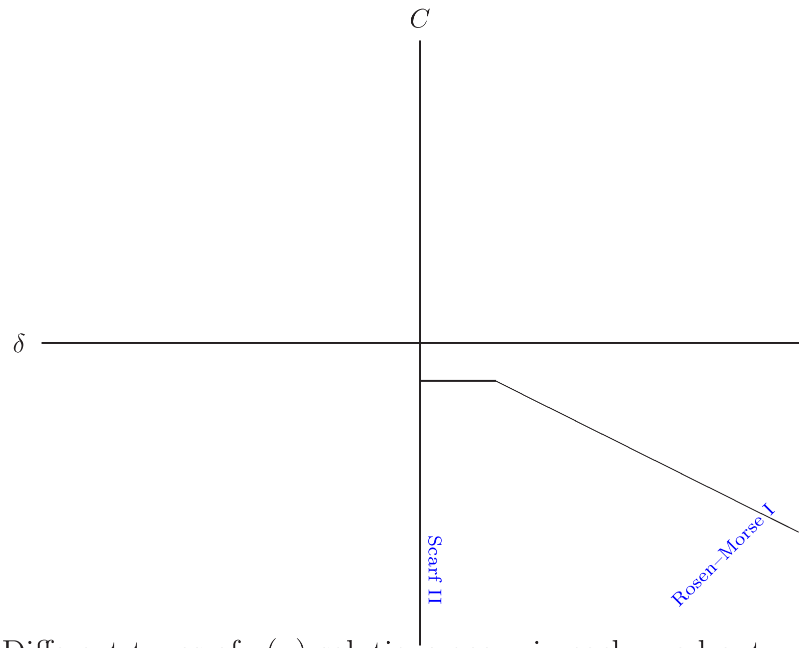
$$(\delta+1)\omega^4 + \delta(2n+1)\omega^3 + \left(\frac{\delta}{4}(2n+1)^2 - \delta - \Sigma - \frac{1}{4}\right)\omega^2 + \Lambda^2 = 0$$

The extrema of these curves can be determined exactly



$V_R(x)$ and $V_I(x)$ for $C = -1$, $\Sigma = 11.0624$ and $\Lambda = 1.26$. The $\delta \to \infty$ limit is the Rosen–Morse I potential. (Note the different scales.) Resembles the $\mathcal{PT}$ square well.

Partial map of the (δ, C) plane

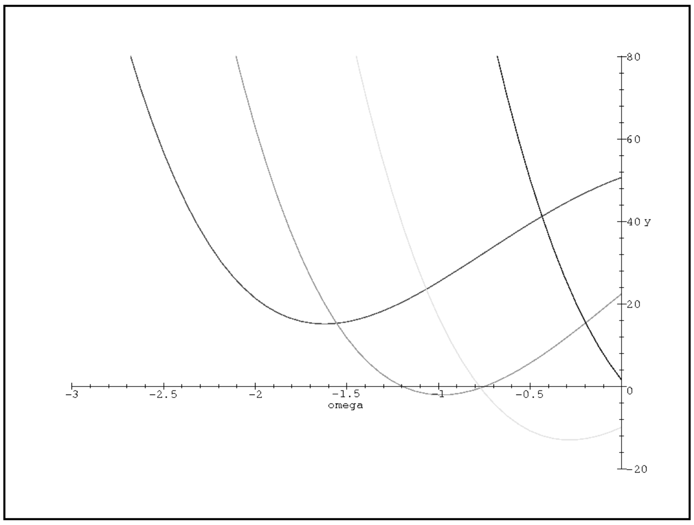


Different types of $z(x)$ solutions occur in each quadrant

The axes are **impenetrable**

Are there complex conjugate ω roots?

$C = -1$, $\delta = 1.25$, $\Sigma = 15.1$, $\Lambda = 7.4$



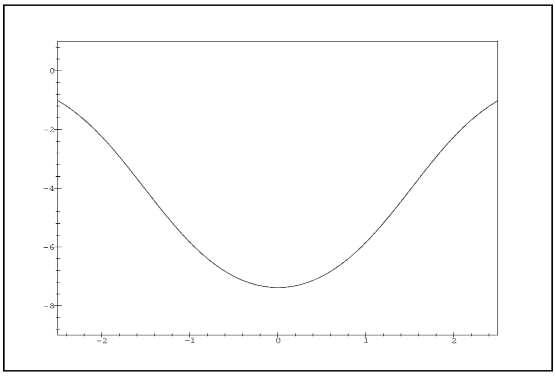
The shifted spectral equation $(\omega + n + 1/2 < 0)$ for $C = -1$, $\delta = 1.25$, $\Sigma = 15.1$, $\Lambda = 7.4$

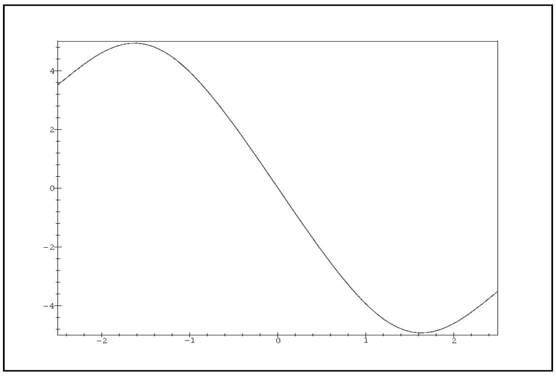
The curves belong to $n = 0$, 1, 2 and 3, counting from the left

Increasing Λ shifts the curves upwards.

The roots then turn into complex starting from the left, i.e. small n.

$\mathrm{Re}(V(x))$ **and** $\mathrm{Im}(V(x))$ **for** $C=-1$**,** $\delta=1.25$**,** $\Sigma=15.1$**,** $\Lambda=7.4$





Energy eigenvalues:

$n=0$	$E_0^{(+)}=-2.555+2.108\mathrm{i}$	$E_0^{(-)}=-2.555-2.108\mathrm{i}$	two complex conjugate eigenvalues
$n=1$	$E_1^{(+)}=-1.439$	$E_1^{(-)}=-0.555$	two real eigenvalues
$n=2$	$E_2^{(+)}=-0.606$		one real eigenvalue

Gradual mechanism for the spontaneous $\mathcal{PT}$ breaking

Starts from below, like in the case of the $\mathcal{PT}$ square well

The $C = a^2 > 0$, $\delta \geq 0$ case: $z(x)$ **real and bounded,** $0 \geq z^2 \geq 1$**,** Λ imaginary

$$C^{1/2}x = \arctan[z(\delta+1-z^2)^{-1/2}] + \delta^{1/2}\text{Artanh}[\delta^{1/2}z(\delta+1-z^2)^{-1/2}] \ .$$

Formally the same as in the $C < 0$ case, because $x(\mathrm{i}z) = \mathrm{i}x(z)$

$\delta \to 0$: $z(x) = \sin(ax)$ Scarf I limit — no complex E_n exist

$\delta \to \infty$: $z(x) = \tanh(\tilde{a}x)$ Rosen–Morse II limit — no complex E_n exist

Now $\mathrm{Im}(V(x))$ does not vanish asymptotically

The wave functions:

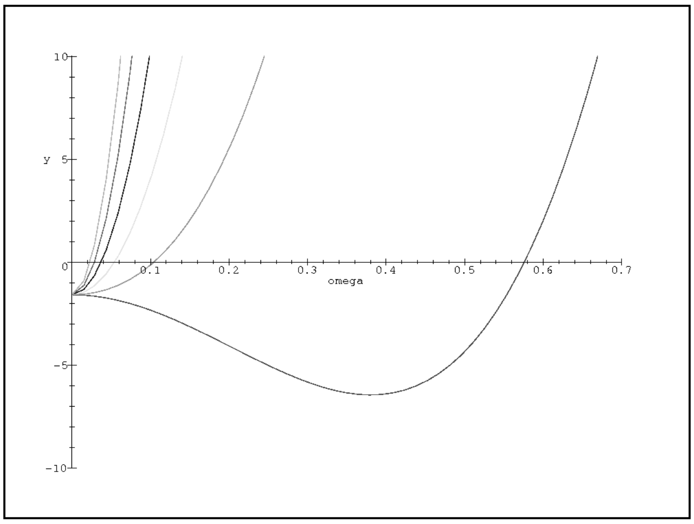
$$\Psi_n(x) = N_n(\delta+1-z^2(x))^{1/4}(1-z(x))^{\alpha/2}(1+z(x))^{\beta/2}P_n^{(\alpha,\beta)}(\mathrm{i}\sinh(x))$$

$$\alpha = \omega + \tfrac{\Lambda}{\omega}, \qquad \beta = \omega - \tfrac{\Lambda}{\omega}$$

Normalizability condition: $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$

Now the roots have to be searched for in the $\omega > 0$ domain.

Are there complex conjugate ω roots?



The spectral equation for $C = 1$, $\delta = 100$, $\Sigma = 11.0624$, $\Lambda = 1.26$ i

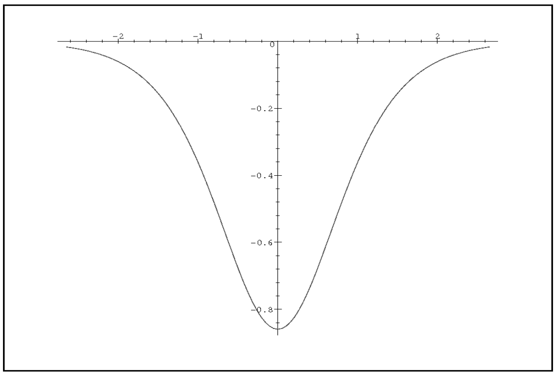
The curves belong to $n = 0$, 1, 2, 3, 4 and 5, counting from the right

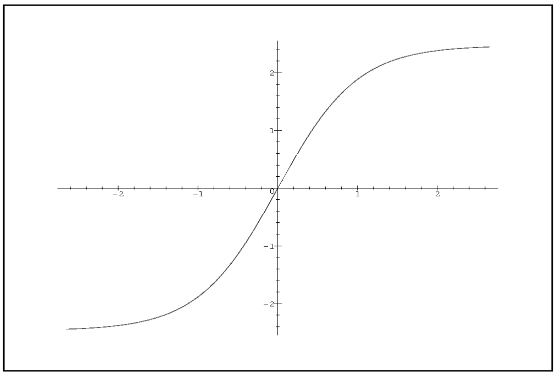
Increasing Λ shifts the curves downwards, including the local maximum at $\omega^{(0)} = 0$.

But $\Pi(\omega^{(+)}) < \Pi(\omega^{(0)})$ $\longrightarrow$ No chance for complex roots.

The descendant potential inherited the properties of its parents...

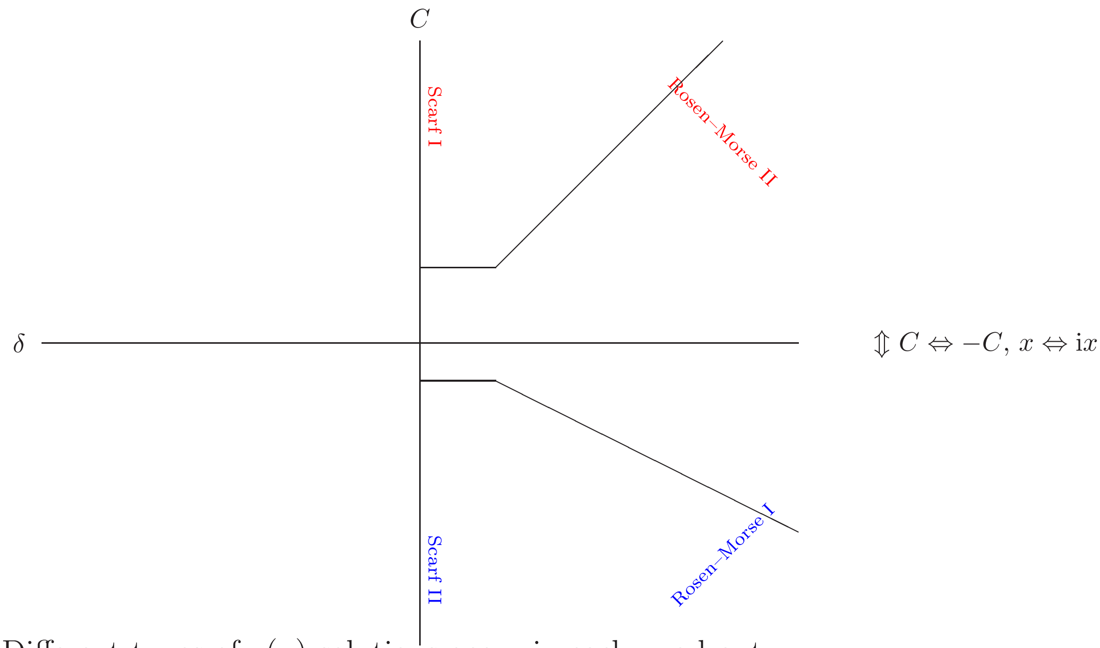
Re$(V(x))$ **and** Im$(V(x))$ **for** $C = 1$, $\delta = 100$, $\Sigma = 11.0624$, $\Lambda = 1.26$ i





The energy eigenvalues are real and positive, starting from $E_0 = 4.454$

A still partial map of the (δ, C) plane



Different types of $z(x)$ solutions occur in each quadrant

The axes are **impenetrable**

The $C = a^2 < 0$, $\delta \leq 0$ case: $z(x)$ real and unbound, $z^2 \geq 1$, Λ imaginary

Now $x(z)$ is different for $\delta < -1$:

$$(-C)^{1/2}x - \mathrm{i}c = \mathrm{Artanh}[z(z^2-1-\delta)^{-1/2}] - \delta^{1/2}\mathrm{Artanh}[(z^2-\delta-1)^{1/2}(-\delta)^{-1/2}z^{-1}] \ .$$

and $-1 < \delta < 0$:

$$(-C)^{1/2}x - \mathrm{i}c = \mathrm{Artanh}[z^{-1}(z^2-1-\delta)^{1/2}] - \delta^{1/2}\mathrm{Artanh}[(-\delta)^{1/2}z(z^2-\delta-1)^{-1/2}] \ .$$

$\delta \to 0$: $\quad z(x) = \cosh(ax) \quad$ generalized Pöschl–Teller limit

$\delta \to \infty$: $\quad z(x) = -\coth(\tilde{a}x) \quad$ Eckart limit

The potentials would be singular without the $x - \mathrm{i}c$ imaginary coordinate shift

$$\Psi_n(x) = N_n(\delta + 1 - z^2(x))^{1/4}(1-z(x))^{\alpha/2}(1+z(x))^{\beta/2}P_n^{(\alpha,\beta)}(\mathrm{i}\sinh(x))$$

$\alpha = \omega + \frac{\Lambda}{\omega}, \qquad \beta = \omega - \frac{\Lambda}{\omega}$

Normalizability condition: $\qquad \mathrm{Re}(\omega) < -n - 1/2$

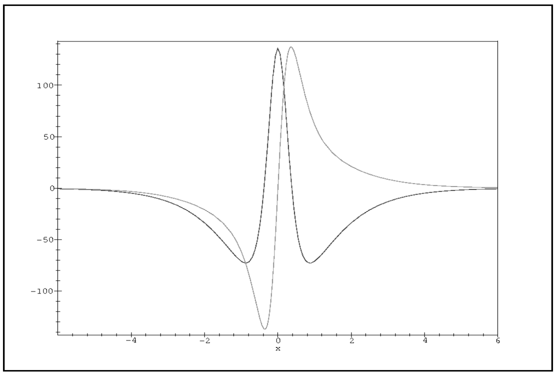
Another special limit: $\quad \delta \to -1 \quad$ the DKV limit

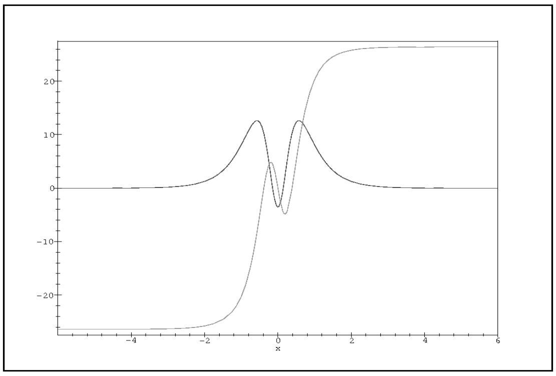
The spectral equation reduces to a cubic one as it should

The complete analysis is missing here.

Complex ω roots and thus complex energy eigenvalues exist here too

We illustrate only the effect of the imaginary coordinate shift for the two limits:





generalized Pöschl–Teller

$\alpha = 3.2, \;\; \beta = -13.2, \;\; c = 0.3$

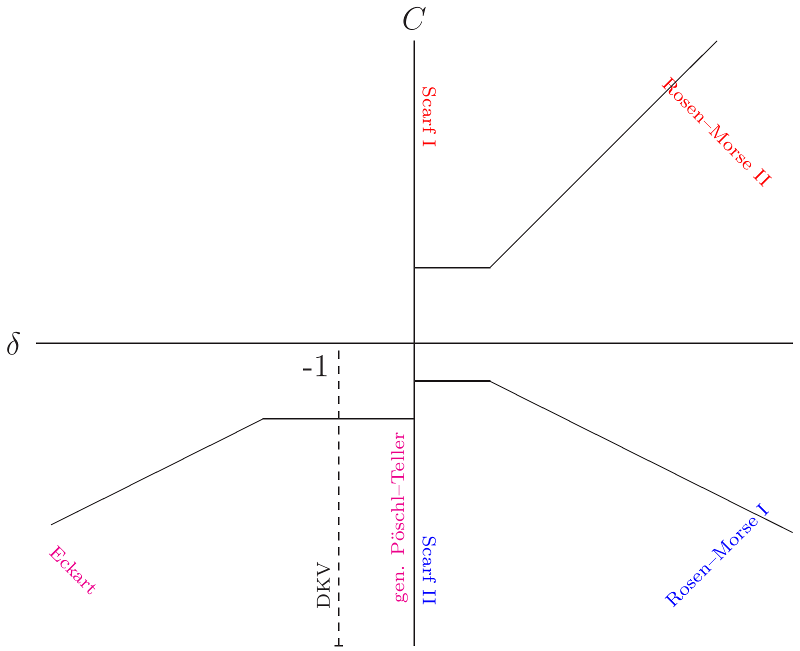
$$E_0 = -20.25$$
$$E_4 = -0.25$$

Eckart

$s = 4.2, \;\; \lambda = 2.6, \;\; c = 0.3$

$$E_0 = -17.2567,$$
$$E_4 = 168.96,$$

A still incomplete map of the (δ, C) plane



$\Updownarrow$ $C \Leftrightarrow -C$, $x \Leftrightarrow \mathrm{i}x$

Different types of $z(x)$ solutions occur in each quadrant

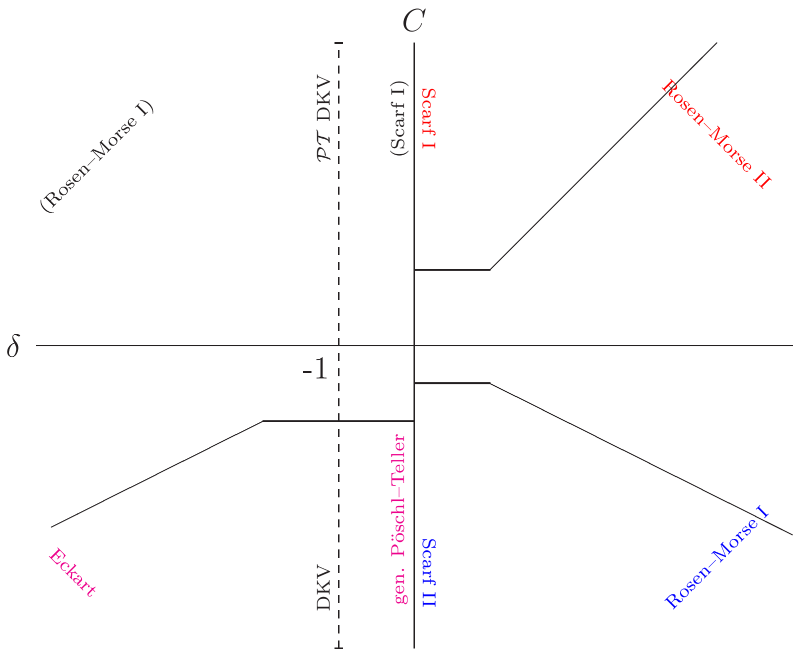
The axes are **impenetrable**

The map of the (δ, C) plane

What about $C > 0$, $\delta \le 0$?

General complex $z(x)$, more complicated situation

However, the limits for $\delta \to 0$, -1 and $-\infty$ are known



$\Updownarrow$ $C \Leftrightarrow -C$, $x \Leftrightarrow \mathrm{i}x$

Discussion. Part 1: the general case

To obtain more flexible spectra we introduced a new potential family

– Implicit $z(x)$, **AND** E_n but tunable spectrum

– It depends on 2+2 parameters and is the subset of the general Natanzon class (3+3)

– It contains all the shape invariant potentials with Jacobi polynomial type solutions

– It also generalizes known Natanzon type potentials (symmetric Ginocchio, DKV)

– Pairs of shape-invariant potentials can directly be connected continuously

Scarf II + Rosen–Morse I, Scarf I + Rosen–Morse II, gen. Pöschl–Teller + Eckart

– In some limits it approximates the finite square well

Discussion. Part 2: the $\mathcal{PT}$-symmetric case

– For $C < 0$, $\delta > 0$ spontaneous $\mathcal{PT}$ breakdown occurs gradually

like the BB potential for $\delta < 0$
with the difference that complex E_n appears from low n

– For $C > 0$, $\delta > 0$ spontaneous $\mathcal{PT}$ breakdown does not occur

Too strong non-hermiticity may not let complex eigenvalues develop

like the BB potential for $\delta > 0$

(non-Hermiticity is like wine)

– For $C < 0$, $\delta < 0$ the problem has to be defined off the real x axis

like the BB potential for $\delta > 2$

– For $C > 0$, $\delta < 0$ only the special limits have been explored

– Scattering solutions yet to be studied in the relevant cases