## Complex Classical Mechanics of a QES system



## Bhabani Prasad Mandal

## Banaras Hindu University, INDIA

Refs:

1. A. Khare \& BPM , PLA 272, 53 (2000).
2. Complex Classical Mechanics of a QES system S. Mahajan\& BPM.

## Plan of the Talk

- Introduction
- Complex Classical Mechanics
- The QES System
- DSHG \& DSG potentials
- Complex Classical Mechanics of QES system
- Conclusions


## Introduction

Huge success of complex quantum theory leads further developments in mainly two major directions;

- Technics of complex quantum theory are used in different branches to get insight of the theory as,
- Quantum Optics: Experimental observation of PT symmetry breaking; Interesting properties of PT material.
- Open Quantum System: Environmentally induced effects in quantum systems are studied through techniques of PT symmetric non-Hermitian systems.
- Quacy-Exactly Solvable System: PT Symmetric non-Hermitian QES systems have been studied extensively.
- Information Theory: State discrimination for pure states (Bender et. al arXiv:1011.1871) and entangled states (A.Ghatak \& B.P.M, JPA 45, 2012).


## Introduction

- Complex extensions of other branches of physics,
- QFT: $i \phi^{3}$ theory, complex QED etc have been studied in details.
$C$-operator has been calculated.
Sign of $\beta$-functions gets reversed, hence the theories which were not asymptotically free, becomes asymptotically free, theories which lacks stable critical points develop such point.
- Many Particle System: Calogero $A_{N-1}, B_{N}$ models have been extend with PT-symmetry non-Hermitian term, retaining their exact solvability and integrability (in certain models).
- Complex Classical Mechanics (CCM): Correspondence principle become more pronounced in the complex domain. Particle shows tunneling like effects.


## Complex Classical Mechanics (CCM)

- Strong analogies between the probabilistic behavior of quantum system defined by Hermitian Hamiltonian and the deterministic behavior of classical system extended into complex domain.
- With complex energy a classical particle can travel from one allowed region to another allowed region separated by classically forbidden path.
- A classical particle with complex energy can tunnel.
- Correspondence between classical and quantum systems is better understood in the complex domain.


## Refs:

C.M. Bender et.al, Phys.Rev.Lett.104:061601(2010)

## CCM

- Correspondence between complex quantum mechanics and complex classical mechanics is established [Mostly numerical studies].
- Quantum probability density matches with the complex classical probability density for high energies.



## The QES System

$$
\begin{equation*}
H=p^{2}-(\zeta \cosh 2 x-i M)^{2} \tag{1}
\end{equation*}
$$

$H$ is invariant under combined Parity ( P ) and Time Reversal ( T )

$$
\begin{aligned}
& \mathbf{P}: x \longrightarrow i \frac{\pi}{2}-x ; p \longrightarrow-p \\
& \mathbf{T}: t \longrightarrow-t ; \quad i \longrightarrow-i ; p \longrightarrow-p
\end{aligned}
$$

QES solutions:
For $M=1$
(2)

$$
E=1-\zeta^{2} ; \quad \phi(x)=\text { constant }
$$

the notation for the wavefunction,

$$
\begin{equation*}
\psi(x)=\phi(x) \exp \frac{i \zeta}{2} \cosh 2 x \tag{3}
\end{equation*}
$$

PT symmetry is unbroken as $i \cosh 2 x$ is invariant under PT.

## The QES System

For $M=2$
(4)

$$
\begin{array}{ll}
E_{+}=3-\zeta^{2}+2 i \zeta ; & \phi(x)=\cosh x \\
E_{-}=3-\zeta^{2}-2 i \zeta ; & \phi(x)=\sinh x
\end{array}
$$

Note PT symmetry is broken spontaneously, as

$$
\begin{array}{r}
\cosh (x) \xrightarrow{P T}-i \sinh (x) \\
\sinh (x) \xrightarrow{P T} i \cosh (x) \tag{5}
\end{array}
$$

## The QES System

For $M=3$
(6)

$$
\begin{aligned}
& E_{0}=5-\zeta^{2} ; \phi(x)=\sinh 2 x \\
& E_{ \pm}=7-\zeta^{2} \pm 2 \sqrt{1-4 \zeta^{2}} ; \phi_{ \pm}=A \cosh 2 x \pm i B
\end{aligned}
$$

With
(7)

$$
\frac{B}{A}=\frac{4 \zeta}{E-9+\zeta^{2}}
$$

We have all three eigenvalues are real for $\zeta \leq \zeta_{c}$. The PT symmetry is unbroken as,

$$
\begin{array}{r}
i \cosh (2 x) \xrightarrow{P T} i \cosh (2 x) \\
\quad \sinh (2 x) \xrightarrow{P T} \sinh (2 x) \tag{8}
\end{array}
$$

## DSG Potential

Anti-isospectral transformation or duality transformation:
If under $x \longrightarrow i x \equiv y$ the potential $v(x) \longrightarrow \bar{v}(y)$, and if the potential $v(x)$ has $M$ QES levels with energy eigenvalue and eigenfunctions
$E_{k}(k=0,1,2 \cdots M-1)$ and $\psi_{k}(x)$ respectively then the energy eigenvalues and eigenfunctions of $\bar{v}(y)$ are given by

$$
\begin{equation*}
\bar{E}_{k}=-E_{M-1-k}, \quad \bar{\psi}_{k}(y)=\psi_{M-1-k}(i x) \tag{9}
\end{equation*}
$$

Under anti-isospectral transformation, $x \longrightarrow i x \equiv \theta$, DSHG potential changes to DSG potential given by

$$
V(\theta)=(\zeta \cos 2 \theta-i M)^{2}
$$

## DSHG \& DSG: Conclusion

For both PT-invariant DSHG and DSG case,

For $M=$ odd integer, $\quad \zeta$ is small

PT symmetry is unbroken and we have real eigenvalues

$$
\text { For } M=\text { even integer, } \quad \zeta \text { is small or large }
$$

PT-symmetry is broken spontaneously and we have pair of complex eigenvalues.

Refs:
Khare \& Mandal, PLA 239 (1998)

## Bender-Dunne Polynomials

- Every QES system is characterized with certain orthogonal polynomial in the variable $E, \quad P_{n}(E)$, which satisfy 3-term recursion relation.
- These polynomials have remarkable properties, for a particular value of the parameter all higher order polynomials factorized to a critical polynomial. Zeros of this critical polynomials are the QES solutions.
- Two sets of BD-polynomials are associated in this particular problem, $P_{n}(E) \& Q_{n}(E)$.
- For $M=2$ critical polynomials are $P_{1}(E) \& Q_{1}(E)$ $M=3$ critical polynomials are $P_{2}(E) \& Q_{1}(E)$
$M=4$ critical polynomials are $P_{2}(E) \& Q_{2}(E)$
$M=5$ critical polynomials are $P_{3}(E) \& Q_{2}(E)$


## Bender-Dunne Polynomials

For $M=2$
(10)

$$
\begin{array}{ll}
E_{+}=3-\zeta^{2}+2 i \zeta ; & \phi(x)=\cosh x \\
E_{-}=3-\zeta^{2}-2 i \zeta ; & \phi(x)=\sinh x
\end{array}
$$

For $M=3$
(11)

$$
E_{0}=5-\zeta^{2} ; \quad \phi(x)=\sinh 2 x
$$

$$
E_{ \pm}=7-\zeta^{2} \pm 2 \sqrt{1-4 \zeta^{2}} ; \phi_{ \pm}=A \cosh 2 x \pm i B
$$

with $\frac{B}{A}=\frac{4 \zeta}{E-9+\zeta^{2}}$

## CCM for QES System

- For a classical particle with real energy $E$ is not allowed to travel in the region where the potential energy $V(x)>E$.
- This restriction is relaxed when we consider the particle in a complex plan with complex energy $E_{1}+i E_{2}$ as

$$
\begin{equation*}
E_{1}=\left(p_{1}^{2}-p_{2}^{2}\right)+V_{1} ; \quad E_{2}=2 p_{1} p_{2}+V_{2} \tag{12}
\end{equation*}
$$

where complex momenta $p=p_{1}+i p_{2}$ and we write the potential $V=V_{1}+i V_{2}$.

- Now since $p_{1}$ and $p_{2}$ can have any value from $-\infty$ to $\infty$ there is no restriction as such on the particle to be bound in a particular region of space.
- Particle is allowed to move anywhere in the complex plane as long as the Eq. (12) is satisfied. This is the prime reason that classical particle with complex energy can travel through classically forbidden region.


## CCM for QES System

- However, even though the particle can exists anywhere in the complex plane, it prefers the region with lower energy.
- Particles follows a definite trajectory depending on initial conditions.
- It has been shown that a classical particle with complex energy executes a local random walk type motion having a open orbit.
- Depending on the value of complex energy classical particles delocalize and move freely in the potential.
- The energy and momentum of the particle are governed by 2 simple classical equations of Hamiltonian mechanics:

$$
\begin{gathered}
\partial H / \partial p=\partial z / \partial t \\
\partial H / \partial z=-\partial p / \partial t
\end{gathered}
$$

## CCM for QES System

- For this system we are considering

$$
\begin{equation*}
\Rightarrow \partial z / \partial t=2 p \quad, \quad \partial p / \partial t=-V^{\prime}(z) \tag{13}
\end{equation*}
$$

where,

$$
V(z)=-(\zeta \cosh 2 z-i M)^{2} .
$$

- We solve these equations numerically to obtain the trajectory in different situations.


## The 1-d Potential



The double well potential for real $x$ and its variation with the parameter $M$ for $\zeta=0.1$

## Potential on the Complex Plane



The potential in complex plane for $\zeta=0.1$ and $M=3$. The potential wells corresponding to the real energy orbits are distributed periodically, centered at $\operatorname{Im}(z)=\frac{4 n+1}{4} \pi$ on the right, and $\operatorname{Im}(z)=\frac{4 n-1}{4} \pi$ on the left of the imaginary axis.

## Position of the Wells



The closed orbits traced by the particle when placed at different places in the complex potential with Real Energy $E=0.8$. This shows us the positions of the wells.

## Tunneling




The trajectory of a particle with energy $E=1+i$ in potential for $\zeta=0.1$;

- when $M=2$, the particle oscillates between the wells corresponding to $n=+3$ on the right $n=-3$ on the left of the imaginary axis,
- when $M=3$, the particle oscillates between the wells corresponding to $n=+10$ on the right and $n=-10$ on the left of the imaginary axis. with tunneling time is $8.2504(R)$ and $8.2513(L)$ units.


## Tunneling



$$
\zeta=0.1, M=4
$$


$\zeta=0.1, M=5$

The trajectory of a particle with energy $E=1+i$ in potential for $\zeta=0.1$;

- when $M=4$, the particle oscillates between the wells corresponding to $n=+21$ on the right and $n=-20$ on the left of the imaginary axis,
- when $M=5$, the particle oscillates between the wells corresponding to $n=+35$ on the right and $n=-35$ on the left of the imaginary axis.


## Tunneling


$\zeta=1, M=2$

$\zeta=1, M=3$

The trajectory of a particle with energy $E=1+i$ in potential for $\zeta=1$;

- when $M=2$, the particle oscillates between the wells corresponding to $n=+3$ on the right and $n=-3$ on the left of the imaginary axis,
- when $M=3$, the particle oscillates between the wells corresponding to $n=+6$ on the right and $n=-6$ on the left of the imaginary axis.


## Tunneling


$\zeta=1, M=4$

$\zeta=1, M=5$

The trajectory of a particle with energy $E=1+i$ in potential for $\zeta=1$;

- for $M=4$, the particle oscillates between the wells corresponding to $n=+11$ on the right and $n=-11$ on the left of the imaginary axis,
- for $M=5$, the particle oscillates between the wells corresponding to $n=+19$ on the right and $n=-19$ on the left of the imaginary axis.


## Tunneling Time Vs. the Imaginary part of Energy



The variation of tunneling time with the imaginary part of energy for a particle initially placed in a potential well corresponding to $n=0$ of PT Symmetric potential ( $M=3, \zeta=0.1$ ) on the right side of the imaginary axis. The real part of energy is fixed as 1 unit.

## Open Orbits with Real Energy


$y+0.4740$


$y+0.52$


Observation: first well $(n=0)$

## Open Orbit with Real Energy



$y+\pi+0.4750$


$y+\pi+0.57$

Observation: second well $(n=1)$

## Real energy= Real part of the potential



$$
y+0.3
$$


$y+0.48$

$y+0.47$

$n=0$, left

## Broken-Unbroken PT


$\zeta=0.1, M=2, E=0.8, y+0.1$

$\zeta=0.1, M=3, E=0.8, y+0.18$

$\zeta=1, M=2, E=0.8, y+0.1$

$\zeta=1, M=3, E=0.8, y+0.1$

Observations:

- $y=.1$ from the center of the zeroth well in the left of imaginary axis with $E=.8$.
- No qualitative difference in PT- unbroken $(\zeta=.1, M=3)$ and PT- broken.


## Conclusions

- We have studied the complex classical mechanics of a system whose non-Hermitian PT-invariant version is QES system

$$
V(x)=-(\zeta \cosh 2 x-i M)^{2} .
$$

- This QES system exhibit very rich behavior and explicitly show the PT-phase transition.
- Where $M=1,3,5 \ldots$ and $\zeta<\zeta_{c}$, it is PT-unbroken phase.
- For $M=2,4, \ldots$ with any $\zeta$ and $M=1,3,5, \ldots$ with high $\zeta$, the system is PT-broken phase.
- We treat this model classically on a complex plane to capture some of the particular behavior of the system.


## Conclusions

- We find that particle tunnels back and forth between two wells (one on the left side and other on the right). Positions of the wells between which it tunnels back and forth depends on the value of $M, \zeta$ and imaginary part of $E$.
- $n$ increases with $M$.
- $n$ decreases when $\zeta$ increases.
- Particle never tunnel between the wells which are located some side of the imaginary axis.
- Time spend by the particle in the left well is different from the time spend in the right well.
- Tunneling time inversely proportional to the imaginary part of the energy. More $E_{I}$ more tunneling effect; Tunneling will be between near wells (less $n$ values).


## Conclusions

- With real energy particle will take a closed path if it is placed sufficiently closed to well.
- Particle with real energy can have open orbit if it is placed initially far away from the well. This can be understood by reducing $E_{I}$ gradually. Very less $E_{I}$ particle will tunnel between wells which are very far.
- We don't observe any difference in the overall behavior of the particle dynamics for PT-broken and unbroken situations. However, the particle trajectory gets a bit irregular PT broken situation.
- same conclusions can also be drawn from the complexified DSG potentials.



## Thanks for Your Attention

