Time-Dependent $\mathcal{PT}$-Symmetric Quantum Mechanics [1]

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Non-Hermitian Operators in Quantum Physics XI
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[1] Work in progress with Jiangbin Gong

Outline

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Schrödinger eqn in conventional QM

- Stationary Schrödinger eqn (eigenvalue eqn)

$$h(t)|\phi_n(t)\rangle = E_n(t)|\phi_n(t)\rangle \tag{1}$$

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- Remarks
  1. Any Hermitian evolution would be unitary.
  2. Eq.(2) cannot be derived from Eq.(1).
  3. The "dual-role" of $h(t)$ is a axiom in QM.

Unitary equivalence

- Stationary Schrödinger eqn is invariant under arbitrary unitary transformations

$$h'(t) = U(t)h(t)U^\dagger(t), \quad |\phi'_n(t)\rangle = U(t)|\phi_n(t)\rangle$$

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- For a time-dependent transformation, the full Schrödinger eqn becomes

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Phi'(t)\rangle = \left[h'(t) - \mathrm{i}\hbar U\dot{U}^\dagger\right]|\Phi'(t)\rangle$$

Inner products

- Inner products in QM [Ballentine, *Quantum Mechanics*]
  1. (ψ, ϕ) is a complex number,
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  3. $(\psi, c_1\phi_1 + c_2\phi_2) = c_1(\psi, \phi_1) + c_2(\psi, \phi_2)$, where c_1 and c_2 are complex numbers,
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- A time-dependent $H(t)$ calls for a time-dependent $W(t)$.

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- Schrödinger-like equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \Lambda(t)|\Psi(t)\rangle$$

with

$$\mathrm{i}\hbar\dot{W} = \Lambda^\dagger W - W\Lambda.$$

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$$H|\psi_n\rangle = E_n|\psi_n\rangle \quad \Rightarrow \quad h|\phi_n\rangle = E_n|\phi_n\rangle$$

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- The lack of the initial condition: Unitary equivalence.

Comments

- For a proper mapping, the evolution equation reduces to

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi\rangle = \left(H - \mathrm{i}\hbar\eta^{-1}\dot{\eta}\right)|\Psi\rangle.$$

[Znojil PRD 2008, SIGMA 2009]

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- Application in conventional QM
  - $W = \mathbf{1}$.
  - $\eta = U$ with $U^{-1} = U^{\dagger}$.
  - If $\eta = \mathbf{1}$ is a proper mapping, then the improper mapping $\eta' = U$ gives

$$\frac{1}{2}\left[\left(\dot{\eta}'\eta'^{-1}\right) - \left(\dot{\eta}'\eta'^{-1}\right)^{\dagger}\right] = \mathrm{i}\hbar\dot{U}U^{\dagger} = -\mathrm{i}\hbar U\dot{U}^{\dagger}.$$

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Complex harmonic oscillator

- Hamiltonian

$$H = \frac{1}{2}\left[\left(X + 2\mathrm{i}\beta\frac{Y}{Z} - \beta^2\frac{Y^2}{Z}\right)\hat{q}^2 + (Y + \mathrm{i}\beta)\left(\hat{p}\hat{q} + \hat{q}\hat{p}\right) + Z\hat{p}^2\right].$$

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- The original $\mathcal{PT}$-symmetric H has the same Berry phase with this h_{GHO}.

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- Instantaneous eigenstates of H

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- Expanding the solution of the Schrödinger-like time evolution Eqn by the complete set,

$$|\Psi\rangle = \sum_n a_n \mathrm{e}^{\mathrm{i}\theta_n}|\psi_n\rangle,$$

where the dynamical phase is $\theta_n(t) = -\frac{1}{\hbar}\int^t \mathrm{d}\tau\, E_n[\mathbf{X}(\tau)]$.

Expansion by instantaneous eigenstates

- Instantaneous eigenstates of H

$$H[\mathbf{X}(t)]|\psi_n(t)\rangle = E_n[\mathbf{X}(t)]|\psi_n[\mathbf{X}(t)]\rangle$$

- Expanding the solution of the Schrödinger-like time evolution Eqn by the complete set,

$$|\Psi\rangle = \sum_n a_n \mathrm{e}^{\mathrm{i}\theta_n}|\psi_n\rangle,$$

where the dynamical phase is $\theta_n(t) = -\frac{1}{\hbar}\int^t \mathrm{d}\tau\, E_n[\mathbf{X}(\tau)]$.

- The time-dependent coefficient,

$$\begin{aligned} \dot{a}_m &= -a_m\left(\langle\psi_m|W|\dot{\psi}_m\rangle + \frac{1}{2}\langle\psi_m|\dot{W}|\psi_m\rangle\right) \\ &\quad + \sum_{n\neq m} a_n\left(\frac{\langle\psi_m|W\dot{H}|\psi_n\rangle}{E_n - E_m} + \frac{1}{2}\langle\psi_m|\dot{W}|\psi_n\rangle\right). \end{aligned}$$

Adiabatic Approximation

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$$\dot{\gamma}_m = \mathrm{i}\left(\langle\psi_m|W|\dot{\psi}_m\rangle + \tfrac{1}{2}\langle\psi_m|\dot{W}|\psi_m\rangle\right).$$

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- Geometry phase in PTQM:

$$\gamma_m^g = \mathrm{i}\int \mathrm{d}\mathbf{X}\cdot\left[\langle\phi_m|W\boldsymbol{\nabla}|\psi_m\rangle + \tfrac{1}{2}\langle\psi_m|(\boldsymbol{\nabla}W)|\psi_m\rangle\right].$$

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2×2 $\mathcal{PT}$-symmetric H

- Hamiltonian

$$
\begin{aligned}
H_{2\times 2} &= e\sigma_0 + \left(a\,\mathbf{n}^r + ib\sin\delta\,\mathbf{n}^\theta + ib\cos\delta\,\mathbf{n}^\varphi\right)\cdot\boldsymbol{\sigma} \\
&= \begin{bmatrix} e + a\cos\theta - ib\sin\theta\sin\delta & (a\sin\theta + ib\cos\theta\sin\delta + b\cos\delta)\mathrm{e}^{-\mathrm{i}\varphi} \\ (a\sin\theta + ib\cos\theta\sin\delta - b\cos\delta)\mathrm{e}^{\mathrm{i}\varphi} & e - a\cos\theta + ib\sin\theta\sin\delta \end{bmatrix}
\end{aligned}
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with

$$
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\mathbf{n}^r &\equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \\
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- Eigenvalues: $E_\pm = e \pm \sqrt{a^2 - b^2}$.

Metric and eigenstates

- Metric

$$W = \mu\left[a\sigma_0 + \left(\nu\,\mathbf{n}^r + b\cos\delta\,\mathbf{n}^\theta - b\sin\delta\,\mathbf{n}^\varphi\right)\cdot\boldsymbol{\sigma}\right],$$

where $a\mu > 0$ and $\nu^2 < a^2 - b^2$.

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$$|\psi_\pm\rangle = \mathcal{N}_\pm \begin{bmatrix} \mathrm{e}^{-\mathrm{i}\varphi}(a\sin\theta + \mathrm{i}b\cos\delta\cos\theta + b\sin\delta) \\ -a\cos\theta + \mathrm{i}b\cos\delta\sin\theta \pm \sqrt{a^2 - b^2} \end{bmatrix}$$

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- The metric we used

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- Geometry phase for $|\psi_\pm\rangle$

$$\gamma_\pm^g = \int \left[F_\pm^\varphi \,\mathrm{d}\varphi + F_\pm^\theta \,\mathrm{d}\theta + F_\pm^\delta \,\mathrm{d}\delta\right],$$

where

$$\begin{aligned} F_\pm^\varphi &= \frac{1}{2}\left(1 \pm \frac{a}{\sqrt{a^2-b^2}}\cos\theta\right) \\ F_\pm^\theta &= \frac{1}{2}\frac{b\sin\delta}{a+b\sin\theta\cos\delta \mp \sqrt{a^2-b^2}\cos\theta} \\ F_\pm^\delta &= \mp\frac{1}{2}\frac{b}{\sqrt{a^2-b^2}}\frac{b+a\sin\theta\cos\delta}{a+b\sin\theta\cos\delta \pm \sqrt{a^2-b^2}\cos\theta}. \end{aligned}$$

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$$\gamma_{\pm}^{B} = \frac{e}{\hbar} \oint \mathbf{A}_{\pm} \cdot \mathrm{d}\mathbf{r} = \frac{e}{\hbar} \iint \boldsymbol{\nabla} \times \mathbf{A}_{\pm} \cdot \mathrm{d}\mathbf{S}.$$

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- In terms of solid angle and winding number,

$$\gamma_{\pm}^{B} = \mp\frac{1}{2}\frac{a}{\sqrt{a^2-b^2}}\Omega + \left(1 \pm \frac{a}{\sqrt{a^2-b^2}}\right)n\pi.$$

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$$\mathbf{B}_\pm = \left(1 \pm \frac{a}{\sqrt{a^2-b^2}}\right)\frac{\pi\hbar}{e}\delta(x)\delta(y)\mathbf{n}^z \mp \frac{a}{\sqrt{a^2-b^2}}\frac{\mathbf{r}}{r^3}\frac{\hbar}{2e}.$$

Improper (Hermitian) mapping

- Two square-roots of W:

$$\eta_\pm = \frac{1}{\chi_\pm\sqrt{2}} \begin{pmatrix} \chi_\pm^2 - b\sin\theta\cos\delta & b(\cos\theta\cos\delta + \mathrm{i}\sin\delta)\mathrm{e}^{-\mathrm{i}\varphi} \\ b(\cos\theta\cos\delta - \mathrm{i}\sin\delta)\mathrm{e}^{\mathrm{i}\varphi} & \chi_\pm^2 + b\sin\theta\cos\delta \end{pmatrix},$$

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$$h_{\pm} \equiv \eta_{\pm} H_{2\times 2} \eta_{\pm}^{-1} = e\sigma_0 \pm \sqrt{a^2 - b^2}\,\mathbf{n}^r \cdot \boldsymbol{\sigma}.$$

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- A proper mapping is $\eta_{\text{proper}} = U^\dagger \eta_\pm$.

Only δ varies

- The eqn for U

$$\frac{\partial U}{\partial \delta} = \mathrm{i}\zeta \mathbf{n}^r \cdot \boldsymbol{\sigma} U, \quad \text{where} \quad \zeta \equiv \mp \frac{b^2}{\sqrt{a^2 - b^2}\left(a \pm \sqrt{a^2 - b^2}\right)}.$$

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- Solution

$$\begin{aligned} U &= \exp(\mathrm{i}\zeta\delta \mathbf{n}^r \cdot \boldsymbol{\sigma}) U_0 \\ &= \begin{pmatrix} \cos(\zeta\delta) + \mathrm{i}\cos\theta\sin(\zeta\delta) & \mathrm{i}\sin\theta\sin(\zeta\delta)\mathrm{e}^{-\mathrm{i}\varphi} \\ \mathrm{i}\sin\theta\sin(\zeta\delta)\mathrm{e}^{\mathrm{i}\varphi} & \cos(\zeta\delta) - \mathrm{i}\cos\theta\sin(\zeta\delta) \end{pmatrix} U_0. \end{aligned}$$

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- Mapped Hamiltonian: $U_0 h_\pm U_0^\dagger$.
- The mapping is a periodic function of δ with period $2\pi/\zeta$.

Only θ varies

- The eqn for U

$$
\begin{aligned}
\frac{\partial U}{\partial \theta} &= \mathrm{i}\zeta\cos\delta \begin{pmatrix} \sin\theta\sin\delta & (-\cos\theta\sin\delta + \mathrm{i}\cos\delta)\mathrm{e}^{-\mathrm{i}\varphi} \\ (-\cos\theta\sin\delta - \mathrm{i}\cos\delta)\mathrm{e}^{\mathrm{i}\varphi} & -\sin\theta\sin\delta \end{pmatrix} U \\
&= -\mathrm{i}\zeta\cos\delta\, \mathrm{e}^{-\mathrm{i}\varphi\sigma_3/2}\mathrm{e}^{-\mathrm{i}\theta\sigma_2/2}\mathrm{e}^{\mathrm{i}\delta\sigma_3/2}\sigma_2\mathrm{e}^{-\mathrm{i}\delta\sigma_3/2}\mathrm{e}^{\mathrm{i}\theta\sigma_2/2}\mathrm{e}^{\mathrm{i}\varphi\sigma_3/2}U.
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\end{aligned}
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$$
U = \mathrm{e}^{-\mathrm{i}\varphi\sigma_3/2}\mathrm{e}^{-\mathrm{i}\theta\sigma_2/2}\exp\left[\mathrm{i}\left(-\zeta\cos\delta\,\mathrm{e}^{\mathrm{i}\delta\sigma_3/2}\sigma_2\mathrm{e}^{-\mathrm{i}\delta\sigma_3/2} + \tfrac{1}{2}\sigma_2\right)\theta\right]U_0.
$$

Only θ varies

- Assume the mapped Hamiltonian has the form

$$h^{\theta} = e\sigma_0 + \sqrt{a^2 - b^2}\begin{pmatrix} \cos\Theta & \sin\Theta\ \mathrm{e}^{-i\Phi} \\ \sin\Theta\ \mathrm{e}^{i\Phi} & -\cos\Theta \end{pmatrix}.$$

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  1. It is independent of φ.

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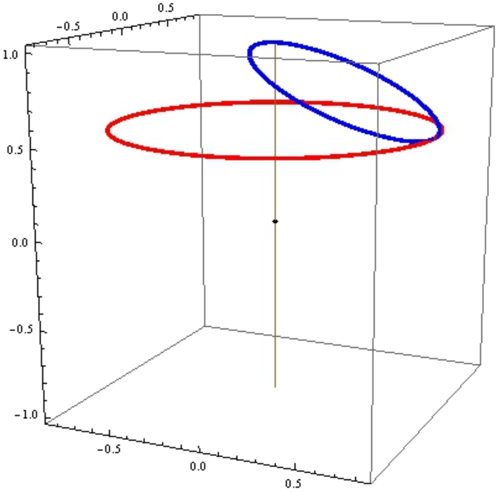
- Remarks
  1. It is independent of φ.
  2. Periodicity of θ is changed.
  3. Multiple-valued function for Θ & Φ.

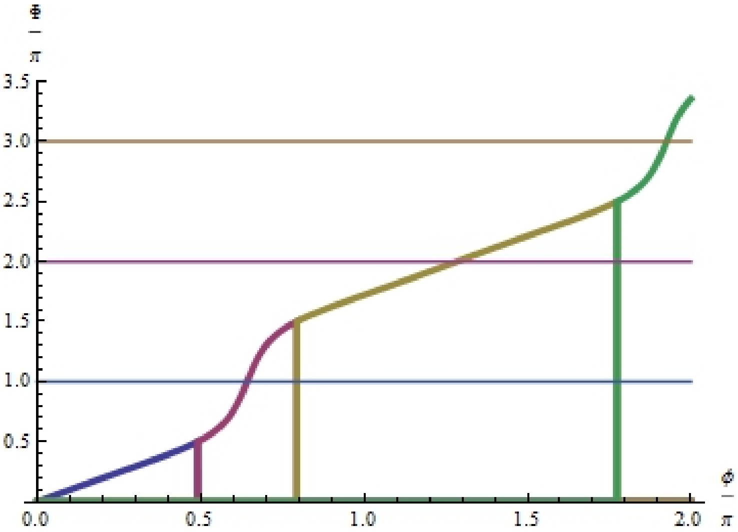
Only φ varies

The red line is a closed path in (θ, φ) space with $\zeta = 0.8$, $\theta = 1.05$, & $\varphi : 0 \to 2\pi$. The blue line is the mapped curve in (Θ, Φ) space.



Only φ varies

Φ as a function of φ for $\zeta = 0.8$, $\theta = 1.05$, & $\varphi : 0 \to 2\pi$.



Outline

Conclusions

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- Finding the proper map may be non-trivial.

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- Conventional full Schrödinger eqn is a special case of Schrödinger-like eqn in PTQM.
- A proper map links a PTQM Hamiltonian to a Hermitian Hamiltonian with "dual roles."
- Finding the proper map may be non-trivial.
- In PTQM, a new geometry phase is found. The associated fictitious magnetic field has a fractional and tunable magnetic monopole and an observable Dirac string.

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- Experiments?