Partial inner product spaces, metric operators and generalized hermiticity

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- Non-self-adjoint operators with real spectrum appear in different contexts :
  - PT-symmetric quantum mechanics (Bender et al.)
  - Pseudo-Hermitian quantum mechanics (Mostafazadeh)
  - Three-Hilbert-Space formulation of quantum mechanics (Znojil)
  - Nonlinear pseudo-bosons (Bagarello)
  - Nonlinear supersymmetry
  - ...

  under various names : pseudo-Hermitian, quasi-Hermitian, cryptohermitian operators
- Generic structure : $A^\dagger = GAG^{-1}$, i.e. $A^\dagger$ is similar to A via $G : \mathcal{H} \to \mathcal{K}$, a metric operator, i.e. $G > 0$, thus invertible, with (possibly unbounded) inverse G^{-1}
- Aim of this talk : to study the problem of operator similarity under a metric operator in the framework of partial inner product spaces (PIP-spaces)

- Metric operator : operator $G \in \mathcal{B}(\mathcal{H})$ such that $G > 0$, i.e. $\langle G\xi|\xi\rangle \geqslant 0, \forall\, \xi \in \mathcal{H}$, and $\langle G\xi|\xi\rangle = 0 \Leftrightarrow \xi = 0$
  - $\Rightarrow$ G invertible, G^{-1} densely defined in $\mathcal{H}$, not necessarily bounded G^{-1} bounded $\Rightarrow$ G^{-1} is a metric operator
  - Let G, G_1, G_2 be metric operators. Then
    - (1) $G_1 + G_2$ is a metric operator
    - (2) λG is a metric operator if $\lambda > 0$
    - (3) if G_1 and G_2 commute, their product $G_1 G_2$ is also a metric operator
    - (4) $G^{1/2}, G^{t/2} (0 \leqslant t \leqslant 1)$ are metric operators
- Define $\langle \xi|\eta\rangle_G := \langle G\xi|\eta\rangle, \xi, \eta \in \mathcal{H}$: positive definite inner product on $\mathcal{H}$ with corresponding norm $\|\xi\|_G = \|G^{1/2}\xi\|$ Completion $\Rightarrow$ new Hilbert space $\mathcal{H}_G$ and $\mathcal{H} \subseteq \mathcal{H}_G$
- Conjugate dual space $\mathcal{H}_G^\times$ of $\mathcal{H}_G \simeq$ subspace of $\mathcal{H}$ and $\mathcal{H}_G^\times \equiv \mathcal{H}_{G^{-1}} = D(G^{-1/2})$ with inner product $\langle \xi|\eta\rangle_{G^{-1}} = \langle G^{-1}\xi|\eta\rangle$

$\Rightarrow$ Triplet (RHS) $\quad \mathcal{H}_{G^{-1}} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_G \quad$ (*)

($\hookrightarrow$ = continuous embedding with dense range)

- G^{-1} bounded $\Rightarrow \mathcal{H}_{G^{-1}} = \mathcal{H}_G = \mathcal{H}$ with norms equivalent (but different) to the norm of $\mathcal{H}$

- In the triplet $\mathcal{H}_{G^{-1}} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_G$ (*)
  - $G^{-1/2}$= unitary operator $\mathcal{H}_{G^{-1}} \to \mathcal{H}$ and $\mathcal{H} \to \mathcal{H}_G$
  - $G^{1/2}$= unitary operator $\mathcal{H} \to \mathcal{H}_{G^{-1}}$ and $\mathcal{H}_G \to \mathcal{H}$
- Triplet (*) = central part of the infinite scale of Hilbert spaces built on the powers of $G^{-1/2}$, $V_I := \{\mathcal{H}_n, n \in \mathbb{Z}\}$, where $\mathcal{H}_n = D(G^{-n/2}), n \in \mathbb{N}$, with a norm equivalent to the graph norm, and $\mathcal{H}_{-n} = \mathcal{H}_n^\times$

$$\ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \subset \ldots$$

- Question : what are the end spaces of the scale ?

$$\mathcal{H}_\infty(G^{-1/2}) := \bigcap_{n\in\mathbb{Z}} \mathcal{H}_n, \qquad \mathcal{H}_{-\infty}(G^{-1/2}) := \bigcup_{n\in\mathbb{Z}} \mathcal{H}_n.$$

- By quadratic interpolation, build continuous scale $\mathcal{H}_t, 0 \leqslant t \leqslant 1$, between $\mathcal{H}_1$ and $\mathcal{H}$, where $\mathcal{H}_t = D(G^{-t/2})$, with norm $\|\xi\|_t = \|G^{-t/2}\xi\|$
- Define $\mathcal{H}_{-t} = \mathcal{H}_t^\times$ and iterate

  $\Rightarrow$ full continuous scale $V_{\tilde{I}} := \{\mathcal{H}_t, t \in \mathbb{R}\}$: PIP-space

- Easy properties :

  Given $(A, D(A))$= linear operator in $\mathcal{H}$:
  - $D(A)$ dense in $\mathcal{H} \;\Rightarrow\; D(A)$ is dense in $\mathcal{H}_G$
  - A closed in $\mathcal{H}_G \;\Rightarrow$ closed in $\mathcal{H}$
- Definitions
  (1) Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, A, B densely defined linear operators in $\mathcal{H}$, resp. $\mathcal{K}$. A bounded operator $T : \mathcal{H} \to \mathcal{K}$ is called an intertwining operator for A and B if
     (i) $T : D(A) \to D(B)$;
     (ii) $BT\xi = TA\xi, \; \forall \xi \in D(A)$.

  (2) A and B are similar $(A \sim B)$ if there exists an intertwining operator T for A and B with bounded inverse $T^{-1} : \mathcal{K} \to \mathcal{H}$, intertwining for B and A ($\sim$ is an equivalence relation)

  (3) A and B are unitarily equivalent $(A \approx B)$ if $A \sim B$ and $T : \mathcal{H} \to \mathcal{K}$ is unitary
- Properties
  - Let $A \sim B$. Then
    - $TD(A) = D(B)$
    - A closed $\Leftrightarrow$ B closed
    - A^{-1} exists $\Leftrightarrow$ B^{-1} exists; then, $B^{-1} \sim A^{-1}$

- Properties (continued)
  - A, B closed and $A \sim B$. Then
    - $\rho(A) = \rho(B)$
    - $\sigma_p(A) = \sigma_p(B)$ and eigenvectors of A, B are in bijection, with same eigenvalues and same multiplicity :
      $\xi \in D(A)$ eigenvector of $A \ \Rightarrow \ G\xi$ eigenvector of B
      $\eta \in D(B)$ eigenvector of $B \ \Rightarrow \ G^{-1}\eta$ eigenvector of A
    - $\sigma_c(A) = \sigma_c(B)$ and $\sigma_r(A) = \sigma_r(B)$
    - A self-adjoint $\Rightarrow$ B has real spectrum and $\sigma_r(B) = \emptyset$
- Similarity : often too strong!
  - A and B are quasi-similar ($A \dashv B$) if there exists an intertwining operator T for A and B which is invertible, with inverse T^{-1} densely defined (but not necessarily bounded).
  - A and B are weakly quasi-similar ($A \dashv_w B$) if B is closable and one has

$$\text{(ws)} \qquad \langle T\xi | B^* \eta \rangle = \langle TA\xi | \eta \rangle, \ \forall \xi \in D(A), \, \eta \in D(B^*).$$

- Properties
  - One can always suppose that T is a metric operator
  - If B is closed, then $A \dashv B \ \Leftrightarrow \ A \dashv_w B$
  - If B is closable and $A \dashv_w B$, then A is closable

- Relation between adjoints
  - Let $(A, D(A))$ be a closed densely defined operator in $\mathcal{H}$. Put
    $$D(A_G^\star) := \{\eta \in \mathcal{H} : G\eta \in D(A^*),\ A^* G\eta \in D(G^{-1})\}$$
    $$A_G^\star \eta := G^{-1} A^* G\eta, \quad \forall \eta \in D(A_G^\star)$$
    Then $A_G^\star$ is the restriction to $\mathcal{H}$ of the adjoint A_G^* of A in $\mathcal{H}_G$
  - If $D(A_G^\star)$ is dense in $\mathcal{H}$ (not automatic!), then A has a densely defined adjoint A_G^* in $\mathcal{H}_G$ and A is closable in $\mathcal{H}_G$
  - Let A, B be closed and densely defined, and $A \dashv B$ with a metric intertwining operator G. Then
    (i) $A_G^\star$ is densely defined
    (ii) $B_0 := (A_G^\star)^*$ is minimal among the closed operators B satisfying, for fixed A and G, the conditions
    $$G : D(A) \to D(B);$$
    $$BG\xi = GA\xi,\ \forall \xi \in D(A),$$
    (iii) $GD(A)$ is a core for B_0.
- Unbounded intertwining operator

  T^{-1} unbounded $\Rightarrow$ need condition milder than continuity, e.g.

  (w) If $\mathcal{H} \ni \{\zeta_n\}$ is such that $T\zeta_n \to 0$, then $\{\zeta_n\}$ admits a subsequence $\{\zeta_{n_k}\}$ weakly convergent to 0

- Comparison of spectra

  Let A, B be closed and densely defined, and assume $A \dashv B$ with the intertwining operator T. Then:

  - $\sigma_p(A) \subseteq \sigma_p(B)$ and, for every $\lambda \in \sigma_p(A)$, one has $m_A(\lambda) \leqslant m_B(\lambda)$
  - If T satisfies condition (w) and $T(D(A))$ is a core for B, then $\sigma_p(B) \subseteq \sigma(A)$
  - In particular, if A has pure point spectrum, then $\sigma_p(B) = \sigma_p(A)$
  - Assume that $D(B), R(B) \subset D(T^{-1})$, $TD(A)$ is a core for B and that T satisfies condition (w). Then $\rho(A) \subset \rho(B) \cup \sigma_c(B)$.
  - In particular, if $\sigma_c(B) = \emptyset$, then $\rho(A) \subseteq \rho(B)$
  - Assume that T^{-1} is everywhere defined and bounded and $TD(A)$ is a core for B. Then

    $$\sigma_p(A) \subseteq \sigma_p(B) \subseteq \sigma(B) \subseteq \sigma(A).$$

  - Remark :This situation is important for applications : it gives some information on $\sigma(B)$ once $\sigma(A)$ is known. For instance
    - A has a pure point spectrum $\Rightarrow$ B is isospectral to A
    - A self-adjoint and B quasi-similar to A via an intertwining operator T with bounded inverse T^{-1} $\Rightarrow$ B has real spectrum

The LHS generated by metric operators - 1

- Define $\mathcal{M}(\mathcal{H}) := \{\text{all metric operators}\}$ and $\mathcal{I}(\mathcal{H}) := \mathcal{M}(\mathcal{H}) \cup \mathcal{M}(\mathcal{H})^{-1}$
- Natural order on $\mathcal{M}(\mathcal{H})$: $G_1 \preceq G_2 \Leftrightarrow \exists \gamma > 0$ such that $G_2 \leqslant \gamma G_1$ $\Leftrightarrow \mathcal{H}_{G_1} \subset \mathcal{H}_{G_2}$ and the identity is continuous and with dense range.
- Extend this order $\preceq$ to $X, Y \in \mathcal{I}(\mathcal{H})$ by putting

$$X \preceq Y \Leftrightarrow \begin{cases} \exists \gamma > 0 : \ Y \leqslant \gamma X, & \text{if } X, Y \in \mathcal{M}(\mathcal{H}), \\ \text{always}, & \text{if } X \in \mathcal{M}(\mathcal{H})^{-1} \text{ and } Y \in \mathcal{M}(\mathcal{H}), \\ \exists \beta > 0 : X^{-1} \leqslant \beta Y^{-1}, & \text{if } X, Y \in \mathcal{M}(\mathcal{H})^{-1}. \end{cases}$$

- Consequences :

$$G_2^{-1} \preceq G_1^{-1} \Leftrightarrow G_1 \preceq G_2 \ \text{ if } G_1, G_2 \in \mathcal{M}(\mathcal{H})$$

$$G^{-1} \preceq I \preceq G, \quad \forall\, G \in \mathcal{M}(\mathcal{H})$$

$\Rightarrow$ Let $X, Y \in \mathcal{I}(\mathcal{H})$. Then, $X \preceq Y \Leftrightarrow \mathcal{H}_X \hookrightarrow \mathcal{H}_Y$

- We will show that the spaces $\{\mathcal{H}_X ; X \in \mathcal{I}(\mathcal{H})\}$ constitute a lattice of Hilbert spaces (LHS) $V_{\mathcal{I}}$

- Let $\mathcal{O} \subset \mathcal{M}(\mathcal{H})$ be a family of metric operators and assume that

$$\mathcal{D} := \bigcap_{G \in \mathcal{O}} D(G^{-1/2})$$

  is a dense subspace of $\mathcal{H}$
- Every operator $G^{-1} \in \mathcal{O}^{-1}$ is self-adjoint, invertible with bounded inverse $\Rightarrow$ on $\mathcal{D}$, define the graph topology $\mathsf{t}_{\mathcal{O}^{-1}}$ by means of the norms

$$\xi \in \mathcal{D} \mapsto \|G^{-1/2}\xi\|, \quad G \in \mathcal{O}$$

- $\mathcal{D}^\times :=$ conjugate dual of $\mathcal{D}[\mathsf{t}_{\mathcal{O}^{-1}}]$, with strong dual topology $\mathsf{t}^\times_{\mathcal{O}^{-1}}$

$$\Rightarrow \qquad \mathcal{D}[\mathsf{t}_{\mathcal{O}^{-1}}] \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{D}^\times[\mathsf{t}^\times_{\mathcal{O}^{-1}}]$$

  the Rigged Hilbert Space associated to $\mathcal{O}^{-1}$
- Then $\mathcal{O}^{-1}$ generates a canonical lattice of Hilbert spaces (LHS) interpolating between $\mathcal{D}$ and $\mathcal{D}^\times$

The LHS generated by metric operators - 3

- On the family $\{\mathcal{H}_X; X \in \mathcal{O}^{-1}\}$ define the lattice operations as

$$\mathcal{H}_{X\wedge Y} := \mathcal{H}_X \cap \mathcal{H}_Y$$
$$\mathcal{H}_{X\vee Y} := \mathcal{H}_X + \mathcal{H}_Y$$

- The corresponding operators read as

$$X \wedge Y := X \dotplus Y$$
$$X \vee Y := (X^{-1} \dotplus Y^{-1})^{-1}$$

Here $\dotplus$ stands for the form sum and $X, Y \in \mathcal{O}^{-1}$:
given two positive operators, $T := T_1 \dotplus T_2$ is the positive operator associated to the quadratic form $\mathfrak{t} = \mathfrak{t}_1 + \mathfrak{t}_2$

- Both $X \vee Y$ and $X \wedge Y$ are inverses of a metric operator, but they need not belong to $\mathcal{O}^{-1}$

$\Rightarrow$ For $\mathcal{O} = \mathcal{M}(\mathcal{H})$, the corresponding family $\mathcal{M}(\mathcal{H})^{-1}$ is a lattice by itself

- The conjugate duals $\mathcal{H}_{X^{-1}}$constitute a dual lattice as

$$\mathcal{H}_{(X\wedge Y)^{-1}} := \mathcal{H}_{X^{-1}} + \mathcal{H}_{Y^{-1}} \,,$$
$$\mathcal{H}_{(X\vee Y)^{-1}} := \mathcal{H}_{X^{-1}} \cap \mathcal{H}_{Y^{-1}} \,.$$

- Let Σ = minimal set of self-adjoint operators containing $\mathcal{O} \cup \mathcal{O}^{-1}$, stable under inversion and form sums, with the property that $\mathcal{D}$ is dense in every H_Z, $Z \in \Sigma$

$\Rightarrow$ $\mathcal{O}^{-1}$ generates a PIP-space with central Hilbert space $\mathcal{H}$, total space $V_{\mathcal{I}} = \sum_{G \in \mathcal{O}} \mathcal{H}_G$ and "smallest" space $V^{\#} = \mathcal{D}$

- Compatibility : $f\#g \iff \exists\, G \in \Sigma$ such that $f \in \mathcal{H}_G$, $g \in \mathcal{H}_{G^{-1}}$
- Partial inner product : $\langle f|g\rangle = \langle G^{1/2} f | G^{-1/2} g\rangle$

Operators on $V_{\mathcal{I}}$

- $A \in \mathrm{Op}(V_{\mathcal{I}}) \;\Leftrightarrow\;$ j$(A) = \{X, Y) \in \mathcal{I}(\mathcal{H}) \times \mathcal{I}(\mathcal{H})$ s.t. $A : \mathcal{H}_X \to \mathcal{H}_Y$, continuously (i.e. bounded)
- $A \simeq \{A_{YX} : (X, Y) \in$ j$(A)\}$: coherent family of representatives

  i.e., $\mathcal{H}_W \subset \mathcal{H}_X, \mathcal{H}_Y \subset \mathcal{H}_Z \;\Rightarrow\; A_{ZW} = E_{ZY} A_{YX} E_{XW}$ ($E_{..} \simeq$ identity)

- Let $(G, G) \in$ j(A), for some $G \in \mathcal{M}(\mathcal{H})$, i.e. $A : \mathcal{H}_G \to \mathcal{H}_G$, bounded

  Then $\mathrm{B}_0 := G^{1/2} A_{GG} G^{-1/2}$ is bounded on $\mathcal{H}_{G^{-1}}$

  $\Rightarrow$ bounded extension $\mathrm{B} := \overline{\mathrm{B}_0}$ to $\mathcal{H}$

  i.e. $A_{GG} \in \mathcal{B}(\mathcal{H}_G)$ is quasi-similar to $\mathrm{B} \in \mathcal{B}(\mathcal{H})$, that is, $A_{GG} \dashv \mathrm{B}$

- More generally, $(X, Y) \in$ j$(A) \;\Leftrightarrow\; X^{1/2} A Y^{-1/2}$ is bounded in $\mathcal{H}$

(Quasi)-similarity of PIP-space operators - 1

- Given $(G, G) \in \mathrm{j}(A)$, $G \in \mathcal{M}(\mathcal{H})$, consider the restriction of A_{GG} to $\mathcal{H}$:

$$D(\mathsf{A}) = \{\xi \in \mathcal{H} : \; A_{GG}\xi = A\xi \in \mathcal{H}\}$$
$$\mathsf{A}\xi = A\xi \, (= A_{GG}\xi), \quad \xi \in D(\mathsf{A}).$$

$D(\mathsf{A})$ need not be dense in $\mathcal{H}$, unless $\mathsf{A}^{\sharp *} \subset A$, where
$\mathsf{A}^{\sharp} = A^{\times} \restriction D(\mathsf{A}^{\sharp}) := \{\xi \in \mathcal{H} : \; A^{\times}\xi \in \mathcal{H}\}$ (dense since $D(\mathsf{A}^{\sharp}) \supset \mathcal{H}_{G^{-1}}$)

Three cases :

(1) If $(G, G) \in \mathrm{j}(A)$, $G \in \mathcal{M}(\mathcal{H})$, then $\mathsf{B}_0 := G^{1/2} A_{GG} G^{-1/2}$ has a bounded closure B

- Assume $\mathsf{A} := A \restriction D(\mathsf{A})$ is densely defined and closed; define

$$D(\breve{\mathsf{B}}) = \{\xi \in \mathcal{H}_{G^{-1}} = D(G^{-1/2}) : \; G^{-1/2}\xi \in D(\mathsf{A})\}$$
$$\breve{\mathsf{B}}\,\xi = G^{1/2}\mathsf{A}G^{-1/2}\xi, \quad \xi \in D(\breve{\mathsf{B}})$$

$\Rightarrow$ $D(\breve{\mathsf{B}})$ dense and $\breve{\mathsf{B}}$ bounded $\Rightarrow$ $\overline{\breve{\mathsf{B}}} = \mathsf{B}$ and

$\Rightarrow$ $G^{1/2} : D(\mathsf{A}) \to D(\mathsf{B})$ and $\mathsf{B}\, G^{1/2}\eta = G^{1/2}\mathsf{A}\,\eta$, $\forall \eta \in D(\mathsf{A})$, i.e. $\mathsf{A} \dashv \mathsf{B}$

- $\mathsf{B}\, G^{1/2}\eta = G^{1/2} A\,\eta$, $\forall \eta \in \mathcal{H}_G$ and $G^{1/2} : \mathcal{H}_G \to \mathcal{H}$ unitary operator

$\Rightarrow$ A and B are unitarily equivalent (but acting in different Hilbert spaces)

(2) If $(G^{-1}, G^{-1}) \in \mathsf{j}(A)$, $G \in \mathcal{M}(\mathcal{H})$, then $A : \mathcal{H}_{G^{-1}} \to \mathcal{H}_{G^{-1}}$ = densely defined operator in $\mathcal{H}$

- $\mathsf{C} := G^{-1/2} A_{G^{-1}G^{-1}} G^{1/2}$ is bounded and everywhere defined on $\mathcal{H}$

  $\Rightarrow$ C and $A_{G^{-1}G^{-1}}$ are unitarily equivalent (in different Hilbert spaces)
- $\mathcal{H}_{G^{-1}} \subset D(\mathsf{A}) = \{\xi \in \mathcal{H} : \mathsf{A}\xi \in \mathcal{H}\} \Rightarrow D(\mathsf{A})$ is dense in $\mathcal{H}$
- One has $\mathsf{C}G^{-1/2}\xi = G^{-1/2}\mathsf{A}\xi$, $\forall \xi \in D(\mathsf{A})$

  $\Rightarrow$ A is not quasi-similar to C (unless $G^{-1/2}$ is bounded too)
- One has $\mathsf{C}G^{-1/2}\xi = G^{-1/2}\mathsf{A}\xi$, $\forall \xi \in \mathcal{H}_{G^{-1}}$ and $G^{-1/2} : \mathcal{H}_{G^{-1}} \to \mathcal{H}$ is unitary

  $\Rightarrow$ $\mathsf{A} \approx \mathsf{C}$, i.e. $G^{1/2}\mathsf{C}G^{-1/2} = \mathsf{A}$ on $\mathcal{H}_{G^{-1}}$

(3) $A \in \mathrm{Op}(V_{\mathcal{I}})$ symmetric if $A = A^{\times}$

- Possibility of self-adjoint restrictions to $\mathcal{H}$ = candidates for quantum observables
- Example : PIP-space version of the KLMN theorem
- Other possibility : exploit PIP-space structure of $\mathrm{Op}(V_{\mathcal{I}})$

If $A = A^{\times}$, then $(G, G) \in \mathsf{j}(A) \Leftrightarrow (G^{-1}, G^{-1}) \in \mathsf{j}(A)$

- $D(\mathsf{A})$ is dense in $\mathcal{H}$
- B and C are unitarily equivalent to (restrictions of) A
- $\mathsf{A} \dashv \mathsf{B}$

- Conclusion : Every symmetric operator $A \in \mathrm{Op}(V_{\mathcal{I}})$ such that $(G, G) \in \mathrm{j}(A)$, with $G \in \mathcal{M}(\mathcal{H})$, is quasi-similar to a bounded operator
- Problem : the assumption $(G, G) \in \mathrm{j}(A)$ is too strong !
  It implies that A has a bounded self-adjoint restriction to $\mathcal{H}$
- Assume $(G^{-1}, G) \in \mathrm{j}(A)$: then we can apply the KLMN theorem :

  Given $A = A^{\times}$, assume $\exists\, G \in \mathcal{M}(\mathcal{H})$ and $\lambda \in \mathbb{R}$ such that $A - \lambda I$ has an invertible representative $(A - \lambda I)_{GG^{-1}} : \mathcal{H}_{G^{-1}} \to \mathcal{H}_G$

  Then $A_{GG^{-1}}$ has a unique restriction to a selfadjoint operator A in $\mathcal{H}$

  - $D(\mathsf{A}) = \{\xi \in \mathcal{H} : A\xi \in \mathcal{H}\}$, dense
  - $\lambda \notin$ spectrum of A
  - The resolvent $(\mathsf{A} - \lambda)^{-1}$ is compact (trace class, etc.) $\Leftrightarrow$ embedding $\mathcal{H}_{G^{-1}} \to \mathcal{H}_G$ is compact (trace class, etc.)
- Open question : is there any quasi-similarity relation between $A_{GG^{-1}}$ or A and another operator?

- Generalization : given $G_1, G_2 \in \mathcal{M}(\mathcal{H})$, what can be said concerning A if it maps $\mathcal{H}_{G_1}$ into $\mathcal{H}_{G_2}$?
- New notion : semi-similarity
  - $\mathcal{H}, \mathcal{K}_1, \mathcal{K}_2$ Hilbert spaces
  - A closed, densely defined operator $A : \mathcal{K}_1 \to \mathcal{K}_2$
  - B closed, densely defined operator on $\mathcal{H}$

  Then A is semi-similar to B ($A \dashv\!\mid B$) if there exist two bounded operators $T : \mathcal{K}_1 \to \mathcal{H}$ and $S : \mathcal{K}_2 \to \mathcal{H}$ such that [(T, S)= intertwining couple]

  (i) $T : D(A) \to D(B)$

  (ii) $BT\xi = SA\xi, \ \forall\, \xi \in D(A)$

$$
\begin{tikzcd}
 & \mathcal{K}_1 \arrow[dl, "T"'] \arrow[dd, "A"] \\
\mathcal{H} & \\
 & \mathcal{K}_2 \arrow[ul, "S"]
\end{tikzcd}
$$

- $\mathcal{K}_1 = \mathcal{K}_2, S = T \ \Rightarrow$ quasi-similarity : $A \dashv B$

- Assume $\exists\, G_1, G_2 \in \mathcal{M}(\mathcal{H})$ such that $A : \mathcal{H}_{G_1} \to \mathcal{H}_{G_2}$ continuously

  Then
  - $\mathrm{B}_0 := G_2^{1/2} A_{G_2 G_1} G_1^{-1/2}$ has a bounded extension B to $\mathcal{H}$ (its closure)
  - $A_{G_2 G_1} \dashv\vdash \mathrm{B}$, w. r. to intertwining couple $T = G_1^{1/2}, S = G_2^{1/2}$
- Take $A = A^{\times}$ symmetric. Then $A : \mathcal{H}_{G_1} \to \mathcal{H}_{G_2} \;\Rightarrow\; A : \mathcal{H}_{G_2^{-1}} \to \mathcal{H}_{G_1^{-1}}$
- Assume $G_1 \preceq G_2$, that is, $\mathcal{H}_{G_1} \subset \mathcal{H}_{G_2}$:

$$\mathcal{H}_{G_2^{-1}} \hookrightarrow \mathcal{H}_{G_1^{-1}} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{G_1} \hookrightarrow \mathcal{H}_{G_2}$$

  KLMN theorem applies

  Assume $\exists\, \lambda \in \mathbb{R}$ s.t. $A - \lambda I$ has an invertible representative $(A - \lambda I)_{G_2 G_2^{-1}} : \mathcal{H}_{G_2^{-1}} \to \mathcal{H}_{G_2}$

  $\Rightarrow$ $A_{G_2 G_2^{-1}}$ has a unique restriction to a self-adjoint operator A in $\mathcal{H}$

  $\Rightarrow$ $A_{G_2 G_2^{-1}} \dashv\vdash \mathrm{B}$ and $\mathrm{A} \dashv\vdash \mathrm{B}$
- Question: A is self-adjoint, but is the spectrum of B real?
- Reference : J-P. Antoine and C. Trapani, *Partial Inner Product Spaces – Theory and Applications*, Lecture Notes in Mathematics, vol. 1986, Springer, Berlin-Heidelberg, 2009