Deformed (anti) commutation relations

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 $\mathcal{D}M^3$ Organization of the . . . Linear pseudo-... Where do pseudo-... Connections with Non-linear pseudo-... Relation with Pseudo-fermions Application to decay What more? Home Page 44 Page 1 of 44 Go Back Full Screen Close Quit

Paris – 30 august 2012

I. Organization of the talk

1. Linear pseudo-bosons: a mathematical introduction



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 - 1. Linear pseudo-bosons: a mathematical introduction
 - 2. Where do pseudo-bosons appear?



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 - 3. Connections with bosons



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 - 3. Connections with bosons
 - 4. Nonlinear pseudo-bosons: a mathematical introduction



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 - 5. Relation with cryptohermiticity



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 - 6. Pseudo-fermions



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 - 7. Applications to decay in quantum optics
 - 8. Conclusions



11. Linear pseudo-bosons: Some mathematics

Let \mathcal{H} be a given Hilbert space with scalar product $\langle ., . \rangle$ and norm ||.||. Let *a* and *b* be two operators acting on \mathcal{H} and satisfying (Trifonov, 2009)

$$[a, b] = \mathbb{1}, \tag{1}$$

If $b = a^{\dagger}$ then we recover CCR. Recall that a and b cannot both be bounded operators: they cannot be defined in all of \mathcal{H} . For this reason we consider the following

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II. Linear pseudo-bosons: Some mathematics

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If $b = a^{\dagger}$ then we recover CCR. Recall that a and b cannot both be bounded operators: they cannot be defined in all of \mathcal{H} . For this reason we consider the following

Assumption 1.– there exists a non-zero $\varphi_0 \in \mathcal{H}$ such that $a\varphi_0 = 0$ and $\varphi_0 \in D^{\infty}(b) := \cap_{k \ge 0} D(b^k)$.



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Then

$$\varphi_n = \frac{1}{\sqrt{n!}} b^n \varphi_0, \quad n \ge 0, \tag{2}$$

belongs to \mathcal{H} for all $n \geq 0$.

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Let N := ba. Then $\varphi_n \in D(N)$, for all $n \ge 0$, and

$$N\varphi_n = n\varphi_n, \quad n \ge 0.$$
 (3)

Let us now take $\mathcal{N} := N^{\dagger} = a^{\dagger}b^{\dagger} \neq N$. We require that the following holds:

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Assumption 2.– there exists a non-zero $\Psi_0 \in \mathcal{H}$ such that $b^{\dagger}\Psi_0 = 0$ and $\Psi_0 \in D^{\infty}(a^{\dagger}) := \cap_{k \ge 0} D((a^{\dagger})^k)$.

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Under this assumption the following vectors

$$\Psi_n = \frac{1}{\sqrt{n!}} (a^{\dagger})^n \Psi_0, \quad n \ge 0, \tag{4}$$

belong to \mathcal{H} for all $n \geq 0$, and to $D(\mathcal{N})$. Moreover

$$\mathcal{N}\Psi_n = n\Psi_n, \quad n \ge 0. \tag{5}$$

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Example 1: the above natural assumptions are not always satisfied: let $\mathcal{H} = \mathcal{L}^2(\mathbb{R}, d\nu(x)), d\nu(x) = \frac{dx}{1+x^2}, a = ip, b = x$. Then $a\varphi_0(x) = 0$ implies that $\varphi_0(x)$ is constant. Of course $\varphi_0(x) \in \mathcal{H}$ but $b\varphi_0(x) = x\varphi_0(x) \notin \mathcal{H}$. Hence $\varphi_0(x)$ does not belong to $D^{\infty}(b)$ and Assumption 1 is violated.

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Example 2: the trivial case: harmonic oscillator. In this case $\mathcal{H} = \mathcal{L}^2(\mathbb{R}, dx)$, and taking $a = c := \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + x\right)$ and $b = c^{\dagger} = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + x\right)$, $[a, b] = [c, c^{\dagger}] = \mathbb{1}$, we find that $\varphi_0(x) = \Psi_0(x) = \frac{1}{\pi^{1/4}}e^{-x^2/2}$, which satisfies both Assumptions 1 and 2.

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Example 3: [Trifonov] $\mathcal{H} = \mathcal{L}^2(\mathbb{R}, dx)$, $a_s = c + sc^{\dagger}$ and $b_s = sc + (1 + s^2)c^{\dagger}$. Hence $[a_s, b_s] = \mathbb{1}$ for all real *s*. $a_s\varphi_0(x) = 0 \Rightarrow \varphi_0(x) = N_s \exp\left\{-\frac{1}{2}\frac{1+s}{1-s}x^2\right\}$, while $b_s^{\dagger}\Psi_0(x) = 0 \Rightarrow \Psi_0(x) = N'_s \exp\left\{-\frac{1}{2}\frac{1+s+s^2}{1-s+s^2}x^2\right\}$. Both these functions are square integrable if -1 < s < 1. This same condition ensures also that $\varphi_0(x) \in D^{\infty}(b_s)$ and that $\Psi_0(x) \in D^{\infty}(a_s^{\dagger})$: any polynomial multiplied for a gaussian function belongs to $\mathcal{L}^2(\mathbb{R}, dx)$.

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Example 4: (two-dimensional deformation of c and c^{\dagger}) Let $a_{\alpha,\mu} := \alpha c + \frac{\alpha}{\mu} c^{\dagger}$, $b_{\alpha,\mu} := \mu \frac{\alpha^2 - 1}{\alpha} c + \alpha c^{\dagger}$, where α and μ are real constants such that $\alpha, \mu \neq 0$ and $\alpha^2 \neq \mu^2(\alpha^2 - 1)$. Hence $a_{\alpha,\mu}^{\dagger} \neq b_{\alpha,\mu}$ and $[a_{\alpha,\mu}, b_{\alpha,\mu}] = \mathbb{1}$. $a_{\alpha,\mu}\varphi_0(x) = 0$ and $b_{\alpha,\mu}^{\dagger}\Psi_0(x) = 0$ produce $\varphi_0(x) = N_{\alpha,\mu} \exp\left\{-\frac{1}{2}\frac{\mu + 1}{\mu - 1}x^2\right\}$,

and

$$\Psi_0(x) = N'_{\alpha,\mu} \exp\left\{-rac{1}{2}rac{lpha^2+\mu(lpha^2-1)}{lpha^2-\mu(lpha^2-1)}x^2
ight\}.$$

These functions satisfy Assumptions 1 and 2 if $\alpha > 1$ and $1 < \mu < 1 + \frac{1}{\alpha^2 - 1}$.

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$$\langle \Psi_n, \varphi_m \rangle = \delta_{n,m}, \quad \forall n, m \ge 0$$
 (6)

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$$\langle \Psi_n, \varphi_m \rangle = \delta_{n,m}, \quad \forall n, m \ge 0$$
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Moreover, $\forall n \geq 0$ we have $\varphi_n \in D(a)$ and $\Psi_n \in D(b^{\dagger})$, and $a\varphi_n = \sqrt{n} \varphi_{n-1}$, as well as $b^{\dagger} \Psi_n = \sqrt{n} \Psi_{n-1}$.

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Moreover, $\forall n \geq 0$ we have $\varphi_n \in D(a)$ and $\Psi_n \in D(b^{\dagger})$, and $a\varphi_n = \sqrt{n} \varphi_{n-1}$, as well as $b^{\dagger} \Psi_n = \sqrt{n} \Psi_{n-1}$.

Let $\mathcal{F}_{\varphi} := \{\varphi_n, n \ge 0\}$ and $\mathcal{F}_{\Psi} := \{\Psi_n, n \ge 0\}$. Since $\langle \varphi_n, \varphi_k \rangle \neq \delta_{n,k}$, $a^{\dagger}\varphi_n = \sqrt{n+1}\varphi_{n+1}$ is false, in general. For the same reason $b \Psi_n \neq \sqrt{n+1} \Psi_{n+1}$.

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However, the sets \mathcal{F}_{φ} and \mathcal{F}_{Ψ} are biorthogonal and, because of this, the vectors of each set are linearly independent.

Assumption 3.– \mathcal{F}_{φ} and \mathcal{F}_{Ψ} are are complete in \mathcal{H} .

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Then, \mathcal{F}_{φ} and \mathcal{F}_{Ψ} are bases in \mathcal{H} . The resolution of the identity looks now

$$\sum_{n=0}^{\infty} |\varphi_n \rangle \langle \Psi_n| = \sum_{n=0}^{\infty} |\Psi_n \rangle \langle \varphi_n| = \mathbb{1}, \quad (7)$$

where $1\!\!1$ is the identity operator on \mathcal{H} .

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where 1 is the identity operator on \mathcal{H} .

Let further

$$S_{\varphi} = \sum_{n=0}^{\infty} |\varphi_n \rangle \langle \varphi_n|, \quad S_{\Psi} = \sum_{n=0}^{\infty} |\Psi_n \rangle \langle \Psi_n|.$$
(8)

These operators need not to be well defined: for instance the series could be not convergent, or even if they do, they could converge to some unbounded operator, so we have to be careful about domains.

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More rigorously, we introduce an operator S_{φ} acting on a vector $f \in D(S_{\varphi})$ as $S_{\varphi}f = \sum_{n=0}^{\infty} \langle \varphi_n, f \rangle \varphi_n$, and S_{Ψ} , acting on a vector $h \in D(S_{\Psi})$ as $S_{\Psi}h =$ $\sum_{n=0}^{\infty} \langle \Psi_n, h \rangle \Psi_n$. Under Assumption 3, both these operators are densely defined in \mathcal{H} . In particular:

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 $S_{\varphi}\Psi_n = \varphi_n, \qquad S_{\Psi}\varphi_n = \Psi_n,$ for all $n \ge 0$. Then $\Psi_n = (S_{\Psi}S_{\varphi})\Psi_n$ and $\varphi_n = (S_{\varphi}S_{\Psi})\varphi_n$, for all $n \ge 0$. Hence (for bounded S_{φ} and S_{Ψ}):

$$S_{\Psi}S_{\varphi} = S_{\varphi}S_{\Psi} = \mathbb{1} \quad \Rightarrow \quad S_{\Psi} = S_{\varphi}^{-1}.$$
 (9)

More rigorously, we introduce an operator S_{ω} acting on a vector $f \in D(S_{\varphi})$ as $S_{\varphi}f = \sum_{n=0}^{\infty} \langle \varphi_n, f \rangle \varphi_n$, and S_{Ψ} , acting on a vector $h \in D(S_{\Psi})$ as $S_{\Psi}h =$ $\sum_{n=0}^{\infty} \langle \Psi_n, h \rangle \Psi_n$. Under Assumption 3, both these operators are densely defined in \mathcal{H} . In particular:

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$$S_{\Psi}S_{\varphi} = S_{\varphi}S_{\Psi} = \mathbb{1} \quad \Rightarrow \quad S_{\Psi} = S_{\varphi}^{-1}.$$
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Furthermore, we can also check that they are both positive defined and symmetric. In general, however, they are unbounded.



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Assumption 4.– \mathcal{F}_{φ} and \mathcal{F}_{Ψ} are Riesz bases: there exist an o.n. basis $\mathcal{G} = \{g_n, n \ge 0\}$ and two bounded operators X and Y, with bounded inverses, such that

 $\varphi_n = X g_n$, and $\Psi_n = Y g_n$,

for all $n \ge 0$.



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, and $\Psi_n = Y g_n$

for all $n \ge 0$.

In thus case we call our pseudo-bosons *regular*, and both S_{φ} and S_{Ψ} are bounded operators.



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In thus case we call our pseudo-bosons *regular*, and both S_{ω} and S_{Ψ} are bounded operators.

Remark:– Regular pseudo-bosons give rise to Riesz bases. Viceversa: each Riesz basis produce two operators *a* and *b* satisfying all the properties of regular pseudo-bosons.



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 S_{Ψ} and S_{φ} are intertwining operators between non self-adjoint operators:

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 S_{Ψ} and S_{φ} are intertwining operators between non self-adjoint operators:

$$S_{\Psi} N = \mathcal{N} S_{\Psi}$$
 and $N S_{\varphi} = S_{\varphi} \mathcal{N}$. (10)

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$$S_{\Psi} N = \mathcal{N} S_{\Psi}$$
 and $N S_{\varphi} = S_{\varphi} \mathcal{N}$. (10)

Some references:

- F. B., Pseudo-bosons, Riesz bases and coherent states,
- J. Math. Phys., (2010)

F. B., Construction of pseudo-bosons systems, J. Math. Phys., (2010)

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111. Where do pseudo-bosons appear?

III.1. The extended quantum harmonic oscillator

[J. da Providência et al., Non hermitian operators with real spectrum in quantum mechanics, arXiv: quant-ph 0909.3054, [F.B, PLA, 2010]

$$H_{\beta} = \frac{\beta}{2} \left(p^2 + x^2 \right) + i\sqrt{2} p,$$

 $\beta > 0 \text{ and } [x, p] = i.$

Organization of the . . . Linear pseudo-... Where do pseudo-... Connections with ... Non-linear pseudo-... Relation with . . . Pseudo-fermions Application to decay What more? Home Page Title Page • 44 Page 13 of 44 Go Back Full Screen Close Quit

111. Where do pseudo-bosons appear?

III.1. The extended quantum harmonic oscillator

[J. da Providência et al., Non hermitian operators with real spectrum in quantum mechanics, arXiv: quant-ph 0909.3054, [F.B, PLA, 2010]

$$H_{\beta} = \frac{\beta}{2} \left(p^2 + x^2 \right) + i\sqrt{2} p,$$

 $\beta > 0$ and [x, p] = i. Using $a = \frac{1}{\sqrt{2}} \left(x + \frac{d}{dx} \right)$, $a^{\dagger} = \frac{1}{\sqrt{2}} \left(x - \frac{d}{dx} \right)$, $[a, a^{\dagger}] =$ **1**, and $N = a^{\dagger}a$, we can write $H_{\beta} = \beta N + (a - a^{\dagger}) + \frac{\beta}{2}$ **1** which, putting

$$A_eta = a - rac{1}{eta}, \qquad B_eta = a^\dagger + rac{1}{eta}, \qquad \Rightarrow$$

 $H_{\beta} = \beta \left(B_{\beta} A_{\beta} + \gamma_{\beta} \mathbb{1} \right),$ where $\gamma_{\beta} = \frac{2+\beta^2}{2\beta^2}$, $\forall \beta > 0$, $A_{\beta}^{\dagger} \neq B_{\beta}$ and $[A_{\beta}, B_{\beta}] = 0$



Organization of the . . . Assumption 1: find a non zero vector $arphi_0^{(eta)} \in \mathcal{H}$ such Linear pseudo-... that $A_{\beta}\varphi_0^{(\beta)} = 0$ and $\varphi_0^{(\beta)} \in D^{\infty}(B_{\beta})$. Where do pseudo-... Connections with ... Non-linear pseudo-... Relation with . . . Pseudo-fermions Application to decay What more? Home Page Title Page • 44 Page 14 of 44 Go Back Full Screen Close Quit

Assumption 1: find a non zero vector $\varphi_0^{(\beta)} \in \mathcal{H}$ such that $A_\beta \varphi_0^{(\beta)} = 0$ and $\varphi_0^{(\beta)} \in D^\infty(B_\beta)$. $A_\beta \varphi_0^{(\beta)} = 0 \Rightarrow a \varphi_0^{(\beta)} = \frac{1}{\beta} \varphi_0^{(\beta)} \Rightarrow \varphi_0^{(\beta)}$ is a standard coherent state with parameter $\frac{1}{\beta}$:

$$arphi_{0}^{(eta)} = U(eta^{-1})arphi_{0} = e^{-1/2eta^{2}} \sum_{k=0}^{\infty} \frac{eta^{-k}}{\sqrt{k!}} arphi_{k}, \qquad (1)$$

where $a\varphi_0 = 0$, and $U(\beta^{-1}) = e^{\frac{1}{\beta}(a^{\dagger}-a)}$ is the unitary (displacement) operator: $\|\varphi_0^{(\beta)}\| = \|\varphi_0\| = 1$.

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Assumption 1: find a non zero vector $\varphi_0^{(\beta)} \in \mathcal{H}$ such that $A_\beta \varphi_0^{(\beta)} = 0$ and $\varphi_0^{(\beta)} \in D^\infty(B_\beta)$. $A_\beta \varphi_0^{(\beta)} = 0 \Rightarrow a \varphi_0^{(\beta)} = \frac{1}{\beta} \varphi_0^{(\beta)} \Rightarrow \varphi_0^{(\beta)}$ is a standard coherent state with parameter $\frac{1}{\beta}$:

$$\varphi_0^{(\beta)} = U(\beta^{-1})\varphi_0 = e^{-1/2\beta^2} \sum_{k=0}^{\infty} \frac{\beta^{-k}}{\sqrt{k!}} \varphi_k,$$
 (1)

where $a\varphi_0 = 0$, and $U(\beta^{-1}) = e^{\frac{1}{\beta}(a^{\dagger}-a)}$ is the unitary (displacement) operator: $\|\varphi_0^{(\beta)}\| = \|\varphi_0\| = 1$.

Since $||B_{\beta}^{k} \varphi_{0}^{(\beta)}|| \leq k! e^{2/\beta}$, $k \geq 0$, $\varphi_{0}^{(\beta)}$ belongs to the domain of all the powers of B_{β} . As a consequence

$$\varphi_n^{(\beta)} = \frac{1}{\sqrt{n!}} B_\beta^n \varphi_0^{(\beta)}, \qquad (2)$$

is well defined for all $n \ge 0$.

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Assumption 2: $B_{\beta}^{\dagger}\Psi_{0}^{(\beta)} = 0 \Rightarrow \Psi_{0}^{(\beta)} = \varphi_{0}^{(-\beta)} = U(-\beta^{-1})\varphi_{0} = U^{-1}(\beta^{-1})\varphi_{0}$ and $\|(A_{\beta}^{\dagger})^{k}\Psi_{0}^{(\beta)}\| \leq k! e^{2/\beta}$, $k \geq 0$. Hence

$$\Psi_{n}^{(\beta)} = \frac{1}{\sqrt{n!}} (A_{\beta}^{\dagger})^{n} \Psi_{0}^{(\beta)}, \qquad (3)$$

is also well defined for all $n \ge 0$.

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Assumption 2: $B_{\beta}^{\dagger}\Psi_{0}^{(\beta)} = 0 \Rightarrow \Psi_{0}^{(\beta)} = \varphi_{0}^{(-\beta)} = U(-\beta^{-1})\varphi_{0} = U^{-1}(\beta^{-1})\varphi_{0}$ and $\|(A_{\beta}^{\dagger})^{k}\Psi_{0}^{(\beta)}\| \leq k! e^{2/\beta}$, $k \geq 0$. Hence

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is also well defined for all $n \ge 0$.

Calling
$$N_{\beta} = B_{\beta}A_{\beta}$$
 and $\mathcal{N}_{\beta} = N_{\beta}^{\dagger} = A_{\beta}^{\dagger}B_{\beta}^{\dagger}$, since

$$N_{\beta} \varphi_n^{(\beta)} = n \varphi_n^{(\beta)}, \qquad \mathcal{N}_{\beta} \Psi_n^{(\beta)} = n \Psi_n^{(\beta)}, \qquad (4)$$

these vectors above are biorthogonal and the following holds:

$$\left\langle \varphi_{n}^{\left(\beta\right)},\Psi_{m}^{\left(\beta\right)}\right\rangle =\delta_{n,m}\,e^{-2/\beta^{2}}.$$
 (5)

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Are $\mathcal{F}_{\varphi}^{(\beta)}$ and $\mathcal{F}_{\Psi}^{(\beta)}$ Riesz bases?

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Are $\mathcal{F}_{\varphi}^{(\beta)}$ and $\mathcal{F}_{\Psi}^{(\beta)}$ Riesz bases?

No: they are related to an orthonormal basis via the following self-adjoint, unbounded and invertible operator: $V_{\beta} = e^{(a+a^{\dagger})/\beta}$, where $[a, a^{\dagger}] = 1$. More explicitly, we have $\varphi_k^{(\beta)} = e^{-1/\beta^2} V_{\beta} \varphi_k$. and $\Psi_k^{(\beta)} = e^{-1/\beta^2} V_{\beta}^{-1} \varphi_k$, where $\varphi_k = \frac{(a^{\dagger})^k}{\sqrt{k!}} \varphi_0$, and $a\varphi_0 = 0$.



Are $\mathcal{F}_{\varphi}^{(\beta)}$ and $\mathcal{F}_{\Psi}^{(\beta)}$ Riesz bases?

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Moreover, calling $h_{\beta} = \beta(a^{\dagger}a + \gamma_{\beta}\mathbb{1}) = h_{\beta}^{\dagger}$, we have

$$H_{\beta}V_{\beta} = V_{\beta}h_{\beta}: \qquad (6)$$

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Moreover, calling $h_{\!\beta} = eta(a^{\dagger}a + \gamma_{\!\beta} 1\!\!1) = h_{\!\beta}^{\dagger}$, we have

$$H_{\beta}V_{\beta} = V_{\beta}h_{\beta}: \qquad (6)$$

 V_{β} is an intertwining operator between h_{β} and H_{β} .

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III.2. The Swanson hamiltonian

The non self-adjoint hamiltonian is

$$H_ heta=rac{1}{2}\left(p^2+x^2
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 $heta \in \left(-rac{\pi}{4},rac{\pi}{4}
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$$H_ heta=N+rac{i}{2} an(2 heta)\left(a^2+(a^\dagger)^2
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where $N = a^{\dagger}a$. If



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where $N = a^{\dagger}a$. If

$$\begin{aligned} \mathcal{A}_{\theta} &= \cos(\theta) \, a + i \sin(\theta) \, a^{\dagger} = \frac{1}{\sqrt{2}} \left(e^{i\theta} x + e^{-i\theta} \, \frac{d}{dx} \right), \\ \mathcal{B}_{\theta} &= \cos(\theta) \, a^{\dagger} + i \sin(\theta) \, a \frac{1}{\sqrt{2}} \left(e^{i\theta} x - e^{-i\theta} \, \frac{d}{dx} \right), \end{aligned}$$

then



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where $N = a^{\dagger}a$. If

$$egin{aligned} &A_{ heta} = \cos(heta) \, a + i \sin(heta) \, a^{\dagger} = rac{1}{\sqrt{2}} \left(e^{i heta} x + e^{-i heta} \, rac{d}{dx}
ight), \ &B_{ heta} = \cos(heta) \, a^{\dagger} + i \sin(heta) \, a rac{1}{\sqrt{2}} \left(e^{i heta} x - e^{-i heta} \, rac{d}{dx}
ight), \end{aligned}$$

then

$$H_{\theta} = \omega_{\theta} \left(B_{\theta} A_{\theta} + \frac{1}{2} \mathbb{1} \right), \qquad (7)$$

where $\omega_{\theta} = \frac{1}{\cos(2\theta)}$. We have $A_{\theta}^{\dagger} \neq B_{\theta}$ and $[A_{\theta}, B_{\theta}] = \mathbb{1}$.



$$\begin{aligned} A_{\theta}\varphi_{0}^{(\theta)} &= 0 \Rightarrow \\ \varphi_{0}^{(\theta)}(x) &= N_{1}\exp\left\{-\frac{1}{2}e^{2i\theta}x^{2}\right\}, \end{aligned} (8) \\ B_{\theta}^{\dagger}\Psi_{0}^{(\theta)} &= 0 \Rightarrow \\ \Psi_{0}^{(\theta)}(x) &= N_{2}\exp\left\{-\frac{1}{2}e^{-2i\theta}x^{2}\right\}. \end{aligned} (9)$$

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$$\begin{aligned} \mathcal{A}_{\theta} \varphi_{0}^{(\theta)} &= 0 \Rightarrow \\ \varphi_{0}^{(\theta)}(x) &= N_{1} \exp\left\{-\frac{1}{2} e^{2i\theta} x^{2}\right\}, \end{aligned} \tag{8} \\ \mathcal{B}_{\theta}^{\dagger} \Psi_{0}^{(\theta)} &= 0 \Rightarrow \\ \Psi_{0}^{(\theta)}(x) &= N_{2} \exp\left\{-\frac{1}{2} e^{-2i\theta} x^{2}\right\}. \end{aligned} \tag{9}$$

Since $\Re(e^{\pm 2i\theta}) > 0 \ \forall \theta \in I$, $\Rightarrow \varphi_0^{(\theta)}(x), \Psi_0^{(\theta)}(x) \in \mathcal{L}^2(\mathbb{R})$. If $\theta \notin I$ Assumptions 1 and 2 are violated!

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$$\begin{aligned} \mathcal{A}_{\theta} \varphi_{0}^{(\theta)} &= 0 \Rightarrow \\ \varphi_{0}^{(\theta)}(x) &= N_{1} \exp\left\{-\frac{1}{2} e^{2i\theta} x^{2}\right\}, \\ \mathcal{B}_{\theta}^{\dagger} \Psi_{0}^{(\theta)} &= 0 \Rightarrow \end{aligned}$$
(8)

$$\Psi_0^{(\theta)}(x) = N_2 \exp\left\{-\frac{1}{2} e^{-2i\theta} x^2\right\}.$$
 (9)

Since $\Re(e^{\pm 2i\theta}) > 0 \ \forall \theta \in I$, $\Rightarrow \varphi_0^{(\theta)}(x), \Psi_0^{(\theta)}(x) \in \mathcal{L}^2(\mathbb{R})$. If $\theta \notin I$ Assumptions 1 and 2 are violated! We find:

$$\begin{split} \varphi_n^{(\theta)}(x) &= \frac{1}{\sqrt{n!}} \, B_\theta^n \, \varphi_0^{(\theta)}(x) = \frac{N_1}{\sqrt{2^n \, n!}} \, H_n\left(e^{i\theta}x\right) \, \exp\left\{-\frac{1}{2} \, e^{2i\theta} \, x^2\right\}, \\ \Psi_n^{(\theta)}(x) &= \frac{1}{\sqrt{n!}} \, (A_\theta^\dagger)^n \, \Psi_0^{(\theta)}(x) = \frac{N_2}{\sqrt{2^n \, n!}} \, H_n\left(e^{-i\theta}x\right) \, \exp\left\{-\frac{1}{2} \, e^{-2i\theta} \, x^2\right\}, \end{split}$$

where $H_n(x)$ is the n-th Hermite polynomial.

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$$\|arphi_n^{(heta)}\|^2 = |N_1|^2 \cos\left(rac{\pi}{\cos(2 heta)}
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 $\|\Psi_n^{(heta)}\|^2 = |N_2|^2 \cos\left(rac{\pi}{\cos(2 heta)}
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where P_n is the n-th Legendre polynomial. Hence Assumptions 1 and 2 are satisfied.

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$$\|arphi_n^{(heta)}\|^2 = |N_1|^2 \cos\left(rac{\pi}{\cos(2 heta)}
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ight),$

where P_n is the n-th Legendre polynomial. Hence Assumptions 1 and 2 are satisfied. The biorthogonality of $\mathcal{F}_{\varphi}^{(\theta)} = \{\varphi_n^{(\theta)}(x), n \ge 0\}$ and $\mathcal{F}_{\Psi}^{(\theta)} = \{\Psi_n^{(\theta)}(x), n \ge 0\}$ produces

$$\int_{\mathbb{R}} H_n\left(e^{-i\theta}x\right) H_m\left(e^{-i\theta}x\right) e^{-e^{-2i\theta}x^2} dx = \delta_{n,m}\sqrt{2^{n+m}\pi n! m!}.$$

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ight)\,P_n\left(rac{1}{\cos(2 heta)}
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We still have to check whether the sets $\mathcal{F}_{\phi}^{(\theta)}$ and $\mathcal{F}_{\psi}^{(\theta)}$ are

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$$\|arphi_n^{(heta)}\|^2 = |N_1|^2 \cos\left(rac{\pi}{\cos(2 heta)}
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$$\int_{\mathbb{R}} H_n\left(e^{-i\theta}x\right) H_m\left(e^{-i\theta}x\right) e^{-e^{-2i\theta}x^2} dx = \delta_{n,m}\sqrt{2^{n+m}\pi n! m!}.$$

We still have to check whether the sets $\mathcal{F}_{\phi}^{(\theta)}$ and $\mathcal{F}_{\psi}^{(\theta)}$ are

(i) complete in $\mathcal{L}^2(\mathbb{R})$;

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$$\|arphi_n^{(heta)}\|^2 = |N_1|^2 \cos\left(rac{\pi}{\cos(2 heta)}
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$$\int_{\mathbb{R}} H_n\left(e^{-i\theta}x\right) H_m\left(e^{-i\theta}x\right) e^{-e^{-2i\theta}x^2} dx = \delta_{n,m}\sqrt{2^{n+m}\pi n! m!}.$$

We still have to check whether the sets $\mathcal{F}_{\varphi}^{(heta)}$ and $\mathcal{F}_{\Psi}^{(heta)}$ are

- (i) complete in $\mathcal{L}^2(\mathbb{R})$;
- (ii) Riesz bases.

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Completeness [Kolmogorov and Fomin]: if $\rho(x)$ is a Lebesgue-measurable function which is different from zero almost everywhere (a.e.) in \mathbb{R} and if there exist two positive constants δ , C such that $|\rho(x)| \leq C e^{-\delta|x|}$ a.e. in \mathbb{R} , then the set $\{x^n \rho(x)\}$ is complete in $\mathcal{L}^2(\mathbb{R})$. Therefore, Assumption 3 is satisfied.



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Riesz bases?: we introduce the unbounded, self-adjoint and invertible operator $T_{\theta} = e^{i\frac{\theta}{2}(a^2 - a^{\dagger^2})}$. Then

$$A_{\theta} = T_{\theta} a T_{\theta}^{-1}, \qquad B_{\theta} = T_{\theta} a^{\dagger} T_{\theta}^{-1}.$$
(10)

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 $\mathcal{T}_{ heta}$ is an IO: let $h_{ heta} = \omega_{ heta} \left(a^{\dagger} a + rac{1}{2} \mathbb{1} \right) = h_{ heta}^{\dagger}$, then

$$H_{\theta}T_{\theta} = T_{\theta}h_{\theta}, \qquad T_{\theta}H_{\theta}^{\dagger} = h_{\theta}T_{\theta}, \qquad (11)$$

and $\alpha \in \mathbb{C}$ exists such that

$$\varphi_n^{(\theta)} = \alpha \, T_\theta \, \varphi_n, \quad \text{and} \quad \Psi_n^{(\theta)} = \frac{1}{\overline{\alpha}} \, T_\theta^{-1} \, \varphi_n \quad (12)$$

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 \Rightarrow nor $\mathcal{F}_{\varphi}^{(\theta)}$ neither $\mathcal{F}_{\Psi}^{(\theta)}$ are Riesz bases: our pseudobosons are non-regular. Also, we deduce that $\eta_{\varphi}^{(\theta)} = |\alpha|^2 T_{\theta}^2$ and $\eta_{\Psi}^{(\theta)} = |\alpha|^{-2} T_{\theta}^{-2}$. This is in agreement with the following (formal) computations:

$$\sum_{n=0}^{\infty} \left| arphi_n^{(heta)}
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angle \langle arphi_n
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ight) rac{1}{lpha} \, T_ heta^{-1} = 1\!\!\!1,$$

$$\sum_{n=0}^{\infty} \left| \varphi_n^{(\theta)} \rangle \langle \varphi_n^{(\theta)} \right| = \alpha \, T_{\theta} \left(\sum_{n=0}^{\infty} \left| \varphi_n \right\rangle \langle \varphi_n | \right) (\alpha T_{\theta})^{\dagger} = |\alpha|^2 T_{\theta}^2 = S_{\varphi}^{(\beta)},$$

as well as

$$\sum_{n=0}^{\infty} \left| \Psi_n^{(\theta)} \rangle \langle \Psi_n^{(\theta)} \right| = \frac{1}{\overline{\alpha}} T_{\theta}^{-1} \left(\sum_{n=0}^{\infty} |\varphi_n\rangle \langle \varphi_n| \right) \left(\frac{1}{\overline{\alpha}} T_{\theta}^{-1} \right)^{\dagger} = \\ = |\alpha|^{-2} T_{\theta}^{-2} = S_{\Psi}^{(\beta)}.$$

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1. Landau levels (dim=2)[FB, ST Ali, JP Gazeau, JMP 2010]

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- 1. Landau levels (dim=2)[FB, ST Ali, JP Gazeau, JMP 2010]
- 2. pseudo-hermitian networks [Jin and Song, arxiv 2011] (work in progress)



- 1. Landau levels (dim=2)[FB, ST Ali, JP Gazeau, JMP 2010]
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- D_N type quantum Calogero model [FB, JMAA 2012, submitted]



IV. Connections with bosons

We have considered the following question: *which is the relation between (regular) pseudo-bosons and ordinary bosons?* The answer is given by the following theorems [F. B., J. Phys. A, **44**, 015205 (2011)]:



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Theorem 1 Let a and b be such that [a, b] = 1, and for which Assumptions 1-4 are satisfied. Then an unbounded, densely defined, operator c on \mathcal{H} exists, and a positive bounded operator \mathcal{T} with bounded inverse \mathcal{T}^{-1} , such that $[c, c^{\dagger}] = 1$. Moreover

$$a = T c T^{-1}, \qquad b = T c^{\dagger} T^{-1}.$$
 (1)

Viceversa, given an unbounded, densely defined, operator c on \mathcal{H} satisfying $[c, c^{\dagger}] = \mathbb{1}$ and a positive bounded operator T with bounded inverse T^{-1} , two operators a and b can be introduced for which $[a, b] = \mathbb{1}$, and for which equations (1) and Assumptions 1-4 are satisfied.

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Theorem 2 Let a and b be such that [a, b] = 1, and Assumptions 1-3 (but not 4) hold true. Then two unbounded, densely defined, operators c and R on \mathcal{H} exist, such that $[c, c^{\dagger}] = 1$ and R is positive, self adjoint and with unbounded inverse R^{-1} . Moreover

$$a = RcR^{-1}, \qquad b = Rc^{\dagger}R^{-1},$$
 (2)

and, introducing $\hat{\varphi}_n = \frac{c^{\dagger^n}}{\sqrt{n!}}\hat{\varphi}_0$, $c\varphi_0 = 0$, then $\hat{\varphi}_n \in$ $D(R) \cap D(R^{-1})$, for all $n \geq 0$, and the sets $\{R\hat{\varphi}_n\}$ and $\{R^{-1}\hat{\varphi}_n\}$ are biorthogonal bases of \mathcal{H} . Viceversa, let us consider two unbounded, densely defined, operators c and R on H satisfying $[c, c^{\dagger}] = \mathbb{1}$ with R positive, self-adjoint with unbounded inverse R^{-1} . Suppose that, introduced $\hat{\varphi}_n$ as above, $\hat{\varphi}_n \in$ $D(R) \cap D(R^{-1})$, for all $n \geq 0$, and that the sets $\{R\hat{\varphi}_n\}$ and $\{R^{-1}\hat{\varphi}_n\}$ are biorthogonal bases of \mathcal{H} . Then two operators a and b can be introduced for which $[a, b] = \mathbb{1}$, and for which equations (2) and Assumptions 1-3 (but not 4) are satisfied.



V. Non-linear pseudo-bosons

Limitation of pseudo-bosons: eigenvalues ϵ_n linear in n.



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We use an idea imported from non-linear coherent states:

$$|z\rangle = e^{-|z|^2/2} \sum_{k=0}^{\infty} \frac{z^n}{\sqrt{n!}} \Phi_n$$

becomes

$$\Xi(z):=N(|z|^2)^{-1/2}\sum_{k=0}^{\infty}\frac{z^n}{\sqrt{\epsilon_n!}}\Phi_n,$$

where $\epsilon_n! = \epsilon_1 \cdots \epsilon_n$, with $\epsilon_0! = 1$ and $N(|z|^2)$ a proper normalization (inside a certain domain of convergence).


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Let a and b be operators on \mathcal{H} and $\{\epsilon_n\}$ such that $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \cdots$. Then [F. B., J. Math. Phys., **52**, 063521, (2011)]..



• **p1.** a non zero vector Φ_0 exists in \mathcal{H} such that $a \Phi_0 = 0$ and $\Phi_0 \in D^{\infty}(b)$.

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```
..the triple (a, b, \{\epsilon_n\}) is a family of non-linear regular pseudo-bosons (NLRPB) if:
```

- **p1.** a non zero vector Φ_0 exists in \mathcal{H} such that $a \Phi_0 = 0$ and $\Phi_0 \in D^{\infty}(b)$.
- p2. a non zero vector η_0 exists in \mathcal{H} such that $b^{\dagger} \eta_0 = 0$ and $\eta_0 \in D^{\infty}(a^{\dagger})$.



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- p3. Calling

$$\Phi_n := rac{1}{\sqrt{\epsilon_n!}} \, b^n \, \Phi_0, \qquad \eta_n := rac{1}{\sqrt{\epsilon_n!}} \, a^{\dagger^n} \, \eta_0,$$

we have, for all $n \ge 0$,

$$a \Phi_n = \sqrt{\epsilon_n} \Phi_{n-1}, \qquad b^{\dagger} \eta_n = \sqrt{\epsilon_n} \eta_{n-1}.$$

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$$a \Phi_n = \sqrt{\epsilon_n} \Phi_{n-1}, \qquad b^{\dagger} \eta_n = \sqrt{\epsilon_n} \eta_{n-1}.$$

• **p4.** $\mathcal{F}_{\Phi} = \{\Phi_n, n \ge 0\}$ and $\mathcal{F}_{\eta} = \{\eta_n, n \ge 0\}$ are bases of \mathcal{H} .



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- **p4.** $\mathcal{F}_{\Phi} = \{\Phi_n, n \ge 0\}$ and $\mathcal{F}_{\eta} = \{\eta_n, n \ge 0\}$ are bases of \mathcal{H} .
- **p5.** \mathcal{F}_{Φ} and \mathcal{F}_{η} are Riesz bases of \mathcal{H} .



Let us introduce the following (not self-adjoint) operators:

$$M = ba, \qquad \mathfrak{M} = M^{\dagger} = a^{\dagger}b^{\dagger}.$$
 (1)

Then we can check that $\Phi_n \in D(M) \cap D(b)$, $\eta_n \in D(\mathfrak{M}) \cap D(a^{\dagger})$, and that

$$b \Phi_n = \sqrt{\epsilon_{n+1}} \Phi_{n+1}, \qquad a^{\dagger} \eta_n = \sqrt{\epsilon_{n+1}} \eta_{n+1}, \quad (2)$$

as well as

$$M\Phi_n = \epsilon_n \Phi_n, \qquad \mathfrak{M}\eta_n = \epsilon_n \eta_n,$$
 (3)

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as well as

$$M\Phi_n = \epsilon_n \Phi_n, \qquad \mathfrak{M}\eta_n = \epsilon_n \eta_n,$$
 (3)

Hence, if $\langle \Phi_0, \eta_0 \rangle = 1$,

$$\langle \Phi_n, \eta_m \rangle = \delta_{n,m},$$
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Moreover

$$\sum_{n} |\Phi_n \rangle < \eta_n| = \sum_{n} |\eta_n \rangle < \Phi_n| = \mathbb{1}, \quad (5)$$

while **p5** implies that $S_{\Phi} := \sum_{n} |\Phi_{n} \rangle \langle \Phi_{n}|$ and $S_{\eta} := \sum_{n} |\eta_{n} \rangle \langle \eta_{n}|$ are positive, bounded, invertible and that $S_{\Phi} = S_{\eta}^{-1}$.

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The new fact is that the operators a and b do not, in general, satisfy any simple commutation rule. Indeed, we can check that, for all $n \ge 0$,

$$[a, b]\Phi_n = (\epsilon_{n+1} - \epsilon_n) \Phi_n, \qquad (6)$$

which is different from [a, b] = 1, except if $\epsilon_n = n$. We end this overview mentioning also that M and \mathfrak{M} are connected by an intertwining operator:

$$MS_{\Phi} = S_{\Phi}\mathfrak{M}$$

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VI. Relation with cryptohermiticity

With M. Znojil we have analyzed the connections between NLRPB and CH in JPA, 2011. The starting point is the following

Definition 3 Let us consider two operators H and Θ acting on the Hilbert space \mathcal{H} , with Θ positive and invertible. Let us call H^{\dagger} the adjoint of H in \mathcal{H} with respect to its scalar product and $H^{\ddagger} = \Theta^{-1}H^{\dagger}\Theta$, when this exists. We will say that H is cryptohermitian with respect to Θ (CHwrt Θ) if $H = H^{\ddagger}$.

We will restrict here to Θ and Θ^{-1} bounded. The operators $\Theta^{\pm 1/2}$ are well defined. Hence we can introduce an operator $h := \Theta^{1/2} H \Theta^{-1/2}$. It is easy to check that $h = h^{\dagger}$. Hence the following definition appears natural:



Definition 4 Assume that H is $CHwrt\Theta$, for H and Θ as above. H is well behaved $wrt \Theta$ if h has only discrete eigenvalues ϵ_n , $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, with eigenvectors e_n : $he_n = \epsilon_n e_n$, $n \in \mathbb{N}_0$, and $\mathcal{E} = \{e_n\}$ is a basis of \mathcal{H} .

Useful technical assumptions:

- 1. the multiplicity of each eigenvalue ϵ_n is one.
- 2. We assume $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \ldots$



Definition 4 Assume that H is $CHwrt\Theta$, for H and Θ as above. H is well behaved wrt Θ if h has only discrete eigenvalues ϵ_n , $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, with eigenvectors e_n : $he_n = \epsilon_n e_n$, $n \in \mathbb{N}_0$, and $\mathcal{E} = \{e_n\}$ is a basis of \mathcal{H} .

Useful technical assumptions:

- 1. the multiplicity of each eigenvalue ϵ_n is one.
- 2. We assume $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \ldots$

Theorem 5 Let H be well behaved wrt Θ , where $\Theta, \Theta^{-1} \in B(\mathcal{H})$, and $\Theta = \Theta^{\dagger}$. Then it is possible to introduce two operators a and b on \mathcal{H} , and a sequence of real numbers $\{\epsilon_n, n \in \mathbb{N}_0\}$, such that the triple $(a, b, \{\epsilon_n\})$ is a family of NLRPB. Vice versa, if $(a, b, \{\epsilon_n\})$ is a family of NLRPB, two operators can be introduced, H and Θ , such that $\Theta, \Theta^{-1} \in B(\mathcal{H})$ and $\Theta = \Theta^{\dagger}$, and H is well behaved

wrt Θ.



Consequences:

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Consequences:

1. Formally we have

$$a = \sum_{n=0}^{\infty} \sqrt{\epsilon_n} |\Phi_{n-1}\rangle < \eta_n|, \quad b = \sum_{n=0}^{\infty} \sqrt{\epsilon_{n+1}} |\Phi_{n+1}\rangle < \eta_n|,$$

as well as

$$h = \sum_{n=0}^{\infty} \epsilon_n |e_n> < e_n|$$

$$H=\sum_{n=0}^{\infty}\epsilon_n|\Phi_n><\eta_n|$$

and

,

$$H^{\dagger} = \sum_{n=0}^{\infty} \epsilon_n |\eta_n > < \Phi_n|.$$

In particular *h*, *H* and H^{\dagger} are isospectrals.



2. Even if *h* is not required to be factorizable, because of our construction it turns out that it can be written as $h = b_{\Theta}a_{\Theta}$, where $a_{\Theta} = \Theta^{1/2}a \Theta^{-1/2}$ and $b_{\Theta} = \Theta^{1/2}b \Theta^{-1/2}$. Incidentally, in general $[a_{\Theta}, b_{\Theta}] = \Theta^{1/2}[a, b] \Theta^{-1/2} \neq [a, b]$, but if $[[a, b], \Theta^{1/2}] = 0$, which is the case for pseudobosons. Therefore, at least at a formal level, our construction shows that the hamiltonian *h* can be written in a factorized form.



VII. Pseudo-fermions

[F.B., J. Phys. A, 2012] The CAR are replaced here by the following rules:

 $\{a, b\} = \mathbb{1}, \quad \{a, a\} = 0, \quad \{b, b\} = 0, \quad (1)$

where the relevant situation is when $b \neq a^{\dagger}$. Compared with Assumptions 1-4 for PB, the only assumptions we might need to require now are the following

- **p1.** a non zero vector φ_0 exists in \mathcal{H} such that $a \varphi_0 = 0$.
- **p2.** a non zero vector Ψ_0 exists in \mathcal{H} such that $b^{\dagger} \Psi_0 = 0$.

However, even these two requirements are automatically satisfied, as a consequence of (1):

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In fact, in \mathcal{H} , it is easy to check that the only non-trivial possible choices of *a* and *b* satisfying (1) are the following:

$$a(1)=\left(egin{array}{cc} 0&1\ 0&0\end{array}
ight),\quad b(1)=\left(egin{array}{cc} eta&-eta^2\ 1&-eta\end{array}
ight),$$



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ight),$$

$$a(2)=\left(egin{array}{cc} lpha & 1\ -lpha^2 & -lpha \end{array}
ight), \quad b(2)=\left(egin{array}{cc} 0 & 0\ 1 & 0 \end{array}
ight),$$

with non zero α and β ,

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ight),$$

$$a(2)=\left(egin{array}{cc} lpha & 1\ -lpha^2 & -lpha \end{array}
ight), \quad b(2)=\left(egin{array}{cc} 0 & 0\ 1 & 0 \end{array}
ight),$$

with non zero α and β , or, maybe more interestingly,

$$a(3) = \left(egin{array}{cc} lpha_{11} & lpha_{12} \ -lpha_{11}^2/lpha_{12} & -lpha_{11} \end{array}
ight), \quad b(3) = \left(egin{array}{cc} eta_{11} & eta_{12} \ -eta_{11}^2/eta_{12} & -eta_{11} \end{array}
ight),$$

with $2\alpha_{11}\beta_{11} - \frac{\alpha_{11}^2\beta_{12}}{\alpha_{12}} - \frac{\beta_{11}^2\alpha_{12}}{\beta_{12}} = 1$. For all these choices, it is easy to show that the two non zero vectors φ_0 and Ψ_0 of **p1** and **p2** do exist. This is not surprising, since det(a) = det(b^{\dagger}) = 0.



For instance, if we take $\alpha_{11} = \frac{1}{3}$, $\beta_{11} = \frac{2}{3}$, and $\alpha_{12} = -\beta_{12} = -i$, we find:

$$a(3)=egin{pmatrix} 1/3 & -i\ -i/9 & -1/3 \end{pmatrix}$$
 , $b(3)=egin{pmatrix} 2/3 & i\ 4i/9 & -2/3 \end{pmatrix}$ $arphi_0=lpha\left(egin{array}{c}1\ -i/3 \end{array}
ight)$, $\Psi_0=eta\left(egin{array}{c}1\ -3i/2 \end{array}
ight)$.

It is not difficult to relate lpha and eta in such a way $\langle arphi_0, \Psi_0
angle = 1.$

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$$arphi_1:=barphi_0, \quad \Psi_1=a^\dagger \Psi_0,$$

as well as the non self-adjoint operators

$$N = ba, \quad \mathcal{N} = N^{\dagger} = a^{\dagger}b^{\dagger}.$$
 (3)

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(2)

$$\varphi_1 := b\varphi_0, \quad \Psi_1 = a^{\dagger}\Psi_0, \quad (2)$$

as well as the non self-adjoint operators

$$N = ba, \quad \mathcal{N} = N^{\dagger} = a^{\dagger} b^{\dagger}.$$
 (3)

We further introduce S_{φ} and S_{Ψ} :

$$S_{\varphi}f = \sum_{n=0}^{1} \langle \varphi_n, f \rangle \varphi_n, \quad S_{\Psi}f = \sum_{n=0}^{1} \langle \Psi_n, f \rangle \Psi_n, \quad (4)$$

 $f \in \mathcal{H}$. Hence we get:

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$$\varphi_1 := b\varphi_0, \quad \Psi_1 = a^{\dagger}\Psi_0, \quad (2)$$

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 $f \in \mathcal{H}$. Hence we get:

1.

$$a\varphi_1 = \varphi_0, \quad b^{\dagger} \Psi_1 = \Psi_0.$$
 (5)

$$\varphi_1 := b\varphi_0, \quad \Psi_1 = a^{\dagger}\Psi_0, \tag{2}$$

as well as the non self-adjoint operators

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 (3)

We further introduce S_{φ} and S_{Ψ} :

$$S_{\varphi}f = \sum_{n=0}^{1} \langle \varphi_n, f \rangle \varphi_n, \quad S_{\Psi}f = \sum_{n=0}^{1} \langle \Psi_n, f \rangle \Psi_n, \quad (4)$$

 $f \in \mathcal{H}$. Hence we get:

1.

$$a\varphi_1 = \varphi_0, \quad b^{\dagger} \Psi_1 = \Psi_0.$$
 (5)

2.

$$N\varphi_n = n\varphi_n, \quad \mathcal{N}\Psi_n = n\Psi_n,$$
 (6)

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3. If $\langle arphi_0, \Psi_0
angle = 1$, then

$$\langle arphi_k, \Psi_n
angle = \delta_{k,n},$$

(7)

for k, n = 0, 1.

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3. If $\langle \varphi_0, \Psi_0 \rangle = 1$, then

$$\langle \varphi_k, \Psi_n \rangle = \delta_{k,n},$$

for k, n = 0, 1.

4. S_{φ} and S_{Ψ} are bounded, strictly positive, selfadjoint, and invertible. They satisfy

 $\|S_{\varphi}\| \leq \|\varphi_0\|^2 + \|\varphi_1\|^2$, $\|S_{\Psi}\| \leq \|\Psi_0\|^2 + \|\Psi_1\|^2$,

$$S_{\varphi}\Psi_n = \varphi_n, \qquad S_{\Psi}\varphi_n = \Psi_n,$$
 (8)

for n = 0, 1, as well as $S_{\varphi} = S_{\Psi}^{-1}$ and the following intertwining relations

$$S_{\Psi}N = \mathcal{N}S_{\Psi}, \qquad S_{\varphi}\mathcal{N} = NS_{\varphi}.$$
 (9)

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(7)

(i) N and N behave as fermionic number operators, having eigenvalues 0 and 1;

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(i) N and \mathcal{N} behave as fermionic number operators, having eigenvalues 0 and 1;

(ii) their related eigenvectors are respectively the vec-

tors in $\mathcal{F}_{\varphi} = \{\varphi_0, \varphi_1\}$ and $\mathcal{F}_{\Psi} = \{\Psi_0, \Psi_1\};$



(i) N and N behave as fermionic number operators, having eigenvalues 0 and 1;

(ii) their related eigenvectors are respectively the vectors in $\mathcal{F}_{\varphi} = \{\varphi_0, \varphi_1\}$ and $\mathcal{F}_{\Psi} = \{\Psi_0, \Psi_1\}$; (iii) *a* and *b*[†] are lowering operators for \mathcal{F}_{φ} and \mathcal{F}_{Ψ} respectively;

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(i) N and \mathcal{N} behave as fermionic number operators, having eigenvalues 0 and 1;

(ii) their related eigenvectors are respectively the vectors in $\mathcal{F}_{\varphi} = \{\varphi_0, \varphi_1\}$ and $\mathcal{F}_{\Psi} = \{\Psi_0, \Psi_1\}$; (iii) a and b^{\dagger} are lowering operators for \mathcal{T} and \mathcal{T} .

(iii) a and b^{\dagger} are lowering operators for \mathcal{F}_{φ} and \mathcal{F}_{Ψ} respectively;

(iv) *b* and a^{\dagger} are rising operators for \mathcal{F}_{φ} and \mathcal{F}_{Ψ} respectively;



(i) N and \mathcal{N} behave as fermionic number operators, having eigenvalues 0 and 1;

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(iii) a and b^{\dagger} are lowering operators for \mathcal{F}_{φ} and \mathcal{F}_{Ψ} respectively;

(iv) b and a^{\dagger} are rising operators for \mathcal{F}_{φ} and \mathcal{F}_{Ψ} respectively;

(v) the two sets \mathcal{F}_{φ} and \mathcal{F}_{Ψ} are biorthonormal;

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(ii) their related eigenvectors are respectively the vectors in $\mathcal{F}_{\varphi} = \{\varphi_0, \varphi_1\}$ and $\mathcal{F}_{\Psi} = \{\Psi_0, \Psi_1\}$;

(iii) a and b^{\dagger} are lowering operators for \mathcal{F}_{φ} and \mathcal{F}_{Ψ} respectively;

(iv) *b* and a^{\dagger} are rising operators for \mathcal{F}_{φ} and \mathcal{F}_{Ψ} respectively;

(v) the two sets \mathcal{F}_{φ} and \mathcal{F}_{Ψ} are biorthonormal;

(vi) the very well-behaved operators \mathcal{S}_φ and \mathcal{S}_Ψ maps

 \mathcal{F}_{φ} in \mathcal{F}_{Ψ} and viceversa;

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(i) N and \mathcal{N} behave as fermionic number operators, having eigenvalues 0 and 1;

(ii) their related eigenvectors are respectively the vectors in $\mathcal{F}_{\varphi} = \{\varphi_0, \varphi_1\}$ and $\mathcal{F}_{\Psi} = \{\Psi_0, \Psi_1\}$;

(iii) a and b^{\dagger} are lowering operators for \mathcal{F}_{φ} and \mathcal{F}_{Ψ} respectively;

(iv) b and a^{\dagger} are rising operators for \mathcal{F}_{φ} and \mathcal{F}_{Ψ} respectively;

(v) the two sets \mathcal{F}_{φ} and \mathcal{F}_{Ψ} are biorthonormal;

(vi) the very well-behaved operators S_{φ} and S_{Ψ} maps \mathcal{F}_{φ} in \mathcal{F}_{Ψ} and viceversa;

(vii) S_{φ} and S_{Ψ} intertwine between operators which are not self-adjoint, in the very same way as they do for PB.

The Assumptions 1-4 are automatically satisfied: we get Riesz bases for free, and we don't need to impose conditions on the domains of operators. Also:

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Theorem 6 Let c and $T = T^{\dagger}$ be two operators on \mathcal{H} such that $\{c, c^{\dagger}\} = \mathbb{1}, c^2 = 0$, and T > 0. Then, defining

$$a = T c T^{-1}, \quad b = T c^{\dagger} T^{-1},$$
 (10)

these operators satisfy (1).

Viceversa, given two operators a and b acting on \mathcal{H} , satisfying (1), it is possible to define two operators, c and T, such that $\{c, c^{\dagger}\} = \mathbb{1}$, $c^2 = 0$, $T = T^{\dagger}$ is strictly positive, and (10) holds.

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VIII. Application to decay

The starting point is the Schrödinger equation

$$i\dot{\Psi}(t) = H_{eff}\Psi(t)$$
, with $H_{eff} = \frac{1}{2} \begin{pmatrix} -i\gamma_a & v \\ \overline{v} & -i\gamma_b \end{pmatrix}$,

where $\gamma_a, \gamma_b > 0$ and $v \in \mathbb{C}$, [Ben-Aryeh etc., JPA, 2004; Trifonov etc., JPA, 2007].

VIII.0.1. Schrödinger representation

Putting $\Phi(t) = e^{\Gamma t} \Psi(t)$, $\Gamma = \frac{1}{2}(\gamma_a + \gamma_b)$, we get $i\dot{\Phi}(t) = H\Phi(t)$, where

$$H = i\Gamma \mathbb{1}_2 + H_{eff} = \begin{pmatrix} -i\gamma & v \\ \overline{v} & i\gamma \end{pmatrix}, \quad \Phi(t) = \begin{pmatrix} \Phi_0(t) \\ \Phi_1(t) \end{pmatrix}$$

Here $\gamma = \frac{1}{2}(\gamma_a - \gamma_b)$. Calling $\Omega := |v|^2 - \gamma^2$ we find

$$\left\{ egin{array}{l} \ddot{\Phi}_0(t) = -\Omega \, \Phi_0(t), \ \ddot{\Phi}_1(t) = -\Omega \, \Phi_1(t). \end{array}
ight.$$

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$\Omega = 0$: the functions $\Phi_0(t)$ and $\Phi_1(t)$ are linear in t, so that

$$\Psi(t) = e^{-\Gamma t} \begin{pmatrix} \Phi_0(t) \\ \Phi_1(t) \end{pmatrix} = \begin{pmatrix} e^{-(\gamma_a + \gamma_b)\frac{t}{2}}(A_0 + B_0 t) \\ e^{-(\gamma_a + \gamma_b)\frac{t}{2}}(A_1 + B_1 t) \end{pmatrix}$$

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 $\Omega > 0$. In this case the solution can be written as $\Psi(t) = e^{-(\gamma_a + \gamma_b)\frac{t}{2}} \begin{pmatrix} A_0 \cos(\sqrt{\Omega} t) + B_0 \sin(\sqrt{\Omega} t) \\ A_1 \cos(\sqrt{\Omega} t) + B_1 \sin(\sqrt{\Omega} t) \end{pmatrix}$,



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 $\Omega > 0$. In this case the solution can be written as

$$\Psi(t) = e^{-(\gamma_a + \gamma_b)\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ A_1 \cos(\sqrt{\Omega} t) + B_1 \sin(\sqrt{\Omega} t) \end{pmatrix}$$

 $\Omega < 0. \text{ In this case the solution can be written as}$ $\Psi(t) = e^{-(\gamma_a + \gamma_b)\frac{t}{2}} \begin{pmatrix} A_0 \exp(\sqrt{|\Omega|} t) + B_0 \exp(-\sqrt{|\Omega|} t) \\ A_1 \exp(\sqrt{|\Omega|} t) + B_1 \exp(-\sqrt{|\Omega|} t) \end{pmatrix}.$

Here A_0 , A_1 , B_0 and B_1 are fixed by the initial conditions.

In all cases, when $t \to \infty$, even if in general $\|\Phi(t)\| \nrightarrow$ 0, we find that

$$|\Psi(t)\| o 0$$



VIII.0.2. Heisenberg representation

The eigenvalues of H can be written as $\lambda_{\pm} := \pm \sqrt{\Omega}$, and the eigenstates are

$$\eta_+ = \left(egin{array}{c} rac{1}{\overline{v}} \left(-i oldsymbol{\gamma} + \sqrt{\Omega}
ight) \ 1 \end{array}
ight)$$
 , $\eta_- = \left(egin{array}{c} -rac{1}{\overline{v}} \left(i oldsymbol{\gamma} + \sqrt{\Omega}
ight) \ 1 \end{array}
ight)$

Notice that $\langle \eta_+, \eta_- \rangle = \frac{2\gamma}{|\nu|^2} \left(\gamma - i\sqrt{\Omega} \right)$, which is zero only if $\gamma = 0$ $(H = H^{\dagger})$ or if $\gamma = i\sqrt{\Omega}$ $(H = -H^{\dagger})$. Also, going back to H_{eff}

$$H_{eff} \eta_{\pm} = E_{\pm} \eta_{\pm}, \qquad E_{\pm} = -\frac{i}{2} (\gamma_a + \gamma_b) \pm \sqrt{\Omega}.$$

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$$H_{eff} \eta_{\pm} = E_{\pm} \eta_{\pm}, \qquad E_{\pm} = -\frac{i}{2} (\gamma_a + \gamma_b) \pm \sqrt{\Omega}.$$

It is possible now to introduce two operators *a* and *b*, such that $\{a, b\} = \mathbb{1}, a^2 = b^2 = 0$, and

$$H = \Omega\left(b a - \frac{1}{2}\mathbb{1}\right) = \Omega\left(N - \frac{1}{2}\mathbb{1}\right),$$

where N = b a.

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To recover the same damping we have found in Schrödinger representation, it is natural to consider the time evolution of the number operator N:

$$N_{eff}(t) = e^{iH_{eff}^{\dagger}t}N e^{-iH_{eff}t}$$

which turns out to be

$$N_{eff}(t) = e^{-2\Gamma t} \left(N e^{-i\Omega t} + \mathcal{N} N (1 - e^{-i\Omega t})
ight).$$

Then, if we estimate the norm of $N_{eff}(t)$, it is trivial to deduce that

$$\|N_{eff}(t)\| \leq 3e^{-2\Gamma t}$$

which goes to zero when t diverges. Hence, as expected, we recover damping also in Heisenberg picture.



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Remark:– Larger dimensional examples can also be constructed, see FB, J. Phys. A, submitted.



1. unbounded operators (with M. Znojil [JPhysA 2012] and with C.Trapani and A. Inoue [JMP 2011] and [JMP submitted])



- unbounded operators (with M. Znojil [JPhysA 2012] and with C.Trapani and A. Inoue [JMP 2011] and [JMP submitted])
- 2. pseudo-bosonic quantum field theory: any spinstatistic theorem?



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- 3. more connections with non-hermitian quantum mechanics



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- 3. more connections with non-hermitian quantum mechanics
- 4. bicoherent states and quantization...
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