Deformed (anti) commutation relations

Fabio Bagarello

Dieetcam

Università di Palermo

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8. *Conclusions*

II. Linear pseudo-bosons: Some mathematics

Let $\mathcal{H}$ be a given Hilbert space with scalar product $\langle .,. \rangle$ and norm $\|.\|$. Let a and b be two operators acting on $\mathcal{H}$ and satisfying (Trifonov, 2009)

$$[a, b] = \mathbb{1}, \tag{1}$$

If $b = a^\dagger$ then we recover CCR. Recall that a and b cannot both be bounded operators: they cannot be defined in all of $\mathcal{H}$. For this reason we consider the following

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Assumption 1.– *there exists a non-zero $\varphi_0 \in \mathcal{H}$ such that $a\varphi_0 = 0$ and $\varphi_0 \in D^\infty(b) := \cap_{k\geq 0} D(b^k)$.*

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Then

$$\varphi_n = \frac{1}{\sqrt{n!}}\, b^n\, \varphi_0, \qquad n \geq 0, \tag{2}$$

belongs to $\mathcal{H}$ for all $n \geq 0$.

Let $N := ba$. Then $\varphi_n \in D(N)$, for all $n \geq 0$, and

$$N\varphi_n = n\varphi_n, \quad n \geq 0. \tag{3}$$

Let us now take $\mathcal{N} := N^\dagger = a^\dagger b^\dagger \neq N$. We require that the following holds:

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Under this assumption the following vectors

$$\Psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n \Psi_0, \quad n \geq 0, \tag{4}$$

belong to $\mathcal{H}$ for all $n \geq 0$, and to $D(\mathcal{N})$. Moreover

$$\mathcal{N}\Psi_n = n\Psi_n, \quad n \geq 0. \tag{5}$$

Example 1: the above natural assumptions are not always satisfied: let $\mathcal{H} = \mathcal{L}^2(\mathbb{R}, d\nu(x))$, $d\nu(x) = \frac{dx}{1+x^2}$, $a = ip$, $b = x$. Then $a\varphi_0(x) = 0$ implies that $\varphi_0(x)$ is constant. Of course $\varphi_0(x) \in \mathcal{H}$ but $b\varphi_0(x) = x\varphi_0(x) \notin \mathcal{H}$. Hence $\varphi_0(x)$ does not belong to $D^\infty(b)$ and Assumption 1 is violated.

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Example 2: *the trivial case*: harmonic oscillator. In this case $\mathcal{H} = \mathcal{L}^2(\mathbb{R}, dx)$, and taking $a = c := \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right)$ and $b = c^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right)$, $[a, b] = [c, c^\dagger] = \mathbb{1}$, we find that $\varphi_0(x) = \Psi_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2}$, which satisfies both Assumptions 1 and 2.

Example 3: [Trifonov] $\mathcal{H} = \mathcal{L}^2(\mathbb{R}, dx)$, $a_s = c + sc^\dagger$ and $b_s = sc + (1+s^2)c^\dagger$. Hence $[a_s, b_s] = \mathbb{1}$ for all real s. $a_s\varphi_0(x) = 0 \Rightarrow \varphi_0(x) = N_s \exp\left\{-\frac{1}{2}\frac{1+s}{1-s}x^2\right\}$, while $b_s^\dagger\Psi_0(x) = 0 \Rightarrow \Psi_0(x) = N'_s \exp\left\{-\frac{1}{2}\frac{1+s+s^2}{1-s+s^2}x^2\right\}$. Both these functions are square integrable if $-1 < s < 1$. This same condition ensures also that $\varphi_0(x) \in D^\infty(b_s)$ and that $\Psi_0(x) \in D^\infty(a_s^\dagger)$: any polynomial multiplied for a gaussian function belongs to $\mathcal{L}^2(\mathbb{R}, dx)$.

Example 4: (two-dimensional deformation of c and $c^\dagger$) Let $a_{\alpha,\mu} := \alpha c + \frac{\alpha}{\mu} c^\dagger$, $b_{\alpha,\mu} := \mu \frac{\alpha^2-1}{\alpha} c + \alpha c^\dagger$, where α and μ are real constants such that $\alpha, \mu \neq 0$ and $\alpha^2 \neq \mu^2(\alpha^2-1)$. Hence $a^\dagger_{\alpha,\mu} \neq b_{\alpha,\mu}$ and $[a_{\alpha,\mu}, b_{\alpha,\mu}] = \mathbb{1}$.

$a_{\alpha,\mu}\varphi_0(x) = 0$ and $b^\dagger_{\alpha,\mu}\Psi_0(x) = 0$ produce

$$\varphi_0(x) = N_{\alpha,\mu} \exp\left\{-\frac{1}{2}\frac{\mu+1}{\mu-1}x^2\right\},$$

and

$$\Psi_0(x) = N'_{\alpha,\mu} \exp\left\{-\frac{1}{2}\frac{\alpha^2+\mu(\alpha^2-1)}{\alpha^2-\mu(\alpha^2-1)}x^2\right\}.$$

These functions satisfy Assumptions 1 and 2 if $\alpha > 1$ and $1 < \mu < 1 + \frac{1}{\alpha^2-1}$.

Under Assumptions 1 and 2 we have $\langle \Psi_n, \varphi_m \rangle = \delta_{n,m} \langle \Psi_0, \varphi_0 \rangle$ for all $n, m \geq 0$. Then, if $\langle \Psi_0, \varphi_0 \rangle = 1$,

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Moreover, $\forall n \geq 0$ we have $\varphi_n \in D(a)$ and $\Psi_n \in D(b^\dagger)$, and $a\varphi_n = \sqrt{n}\,\varphi_{n-1}$, as well as $b^\dagger \Psi_n = \sqrt{n}\,\Psi_{n-1}$.

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Let $\mathcal{F}_\varphi := \{\varphi_n,\ n \geq 0\}$ and $\mathcal{F}_\Psi := \{\Psi_n,\ n \geq 0\}$. Since $\langle \varphi_n, \varphi_k \rangle \neq \delta_{n,k}$, $a^\dagger \varphi_n = \sqrt{n+1}\,\varphi_{n+1}$ is false, in general. For the same reason $b\,\Psi_n \neq \sqrt{n+1}\,\Psi_{n+1}$.

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However, the sets $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthogonal and, because of this, the vectors of each set are linearly independent.

Assumption 3.– *$\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are are complete in $\mathcal{H}$.*

Then, $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are bases in $\mathcal{H}$. The resolution of the identity looks now

$$\sum_{n=0}^{\infty} |\varphi_n >< \Psi_n| = \sum_{n=0}^{\infty} |\Psi_n >< \varphi_n| = \mathbb{1}, \qquad (7)$$

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Let further

$$S_\varphi = \sum_{n=0}^{\infty} |\varphi_n >< \varphi_n|, \quad S_\Psi = \sum_{n=0}^{\infty} |\Psi_n >< \Psi_n|. \quad (8)$$

These operators need not to be well defined: for instance the series could be not convergent, or even if they do, they could converge to some unbounded operator, so we have to be careful about domains.

More rigorously, we introduce an operator S_φ acting on a vector $f \in D(S_\varphi)$ as $S_\varphi f = \sum_{n=0}^{\infty} \langle \varphi_n, f \rangle \varphi_n$, and S_Ψ, acting on a vector $h \in D(S_\Psi)$ as $S_\Psi h = \sum_{n=0}^{\infty} \langle \Psi_n, h \rangle \Psi_n$. Under Assumption 3, both these operators are densely defined in $\mathcal{H}$. In particular:

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$$S_\varphi \Psi_n = \varphi_n, \qquad S_\Psi \varphi_n = \Psi_n,$$

for all $n \geq 0$. Then $\Psi_n = (S_\Psi S_\varphi)\Psi_n$ and $\varphi_n = (S_\varphi S_\Psi)\varphi_n$, for all $n \geq 0$. Hence (for bounded S_φ and S_Ψ):

$$S_\Psi S_\varphi = S_\varphi S_\Psi = \mathbb{1} \quad \Rightarrow \quad S_\Psi = S_\varphi^{-1}. \qquad (9)$$

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Furthermore, we can also check that they are both positive defined and symmetric. In general, however, they are unbounded.

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Assumption 4.– *$\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are Riesz bases: there exist an o.n. basis $\mathcal{G} = \{g_n,\ n \geq 0\}$ and two bounded operators X and Y, with bounded inverses, such that*

$$\varphi_n = X\, g_n, \quad \text{and} \quad \Psi_n = Y\, g_n,$$

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Remark:– Regular pseudo-bosons give rise to Riesz bases. Viceversa: each Riesz basis produce two operators a and b satisfying all the properties of regular pseudo-bosons.

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Some references:

F. B., Pseudo-bosons, Riesz bases and coherent states, J. Math. Phys., (2010)

F. B., Construction of pseudo-bosons systems, J. Math. Phys., (2010)

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III. Where do pseudo-bosons appear?

III.1. The extended quantum harmonic oscillator

[J. da Providência et al., Non hermitian operators with real spectrum in quantum mechanics, arXiv: quant-ph 0909.3054, [F.B, PLA, 2010]

$$H_\beta = \frac{\beta}{2}\left(p^2 + x^2\right) + i\sqrt{2}\,p,$$

$\beta > 0$ and $[x, p] = i$.

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$$H_\beta = \frac{\beta}{2}\left(p^2 + x^2\right) + i\sqrt{2}\,p,$$

$\beta > 0$ and $[x, p] = i$.

Using $a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right)$, $a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$, $[a, a^\dagger] = \mathbb{1}$, and $N = a^\dagger a$, we can write $H_\beta = \beta N + (a - a^\dagger) + \frac{\beta}{2}\,\mathbb{1}$ which, putting

$$A_\beta = a - \frac{1}{\beta}, \qquad B_\beta = a^\dagger + \frac{1}{\beta}, \qquad \Rightarrow$$

$$H_\beta = \beta\left(B_\beta A_\beta + \gamma_\beta\,\mathbb{1}\right),$$

where $\gamma_\beta = \frac{2+\beta^2}{2\beta^2}$, $\forall\beta > 0$, $A_\beta^\dagger \neq B_\beta$ and $[A_\beta, B_\beta] = \mathbb{1}$.

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$A_\beta \varphi_0^{(\beta)} = 0 \Rightarrow a\,\varphi_0^{(\beta)} = \frac{1}{\beta}\,\varphi_0^{(\beta)} \Rightarrow \varphi_0^{(\beta)}$ is a standard coherent state with parameter $\frac{1}{\beta}$:

$$\varphi_0^{(\beta)} = U(\beta^{-1})\varphi_0 = e^{-1/2\beta^2} \sum_{k=0}^{\infty} \frac{\beta^{-k}}{\sqrt{k!}}\,\varphi_k, \qquad (1)$$

where $a\varphi_0 = 0$, and $U(\beta^{-1}) = e^{\frac{1}{\beta}(a^\dagger - a)}$ is the unitary (displacement) operator: $\|\varphi_0^{(\beta)}\| = \|\varphi_0\| = 1$.

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Since $\|B_\beta^k\,\varphi_0^{(\beta)}\| \leq k!\,e^{2/\beta}$, $k \geq 0$, $\varphi_0^{(\beta)}$ belongs to the domain of all the powers of B_β. As a consequence

$$\varphi_n^{(\beta)} = \frac{1}{\sqrt{n!}}\,B_\beta^n \varphi_0^{(\beta)}, \qquad (2)$$

is well defined for all $n \geq 0$.

Assumption 2: $B_\beta^\dagger \Psi_0^{(\beta)} = 0 \Rightarrow \Psi_0^{(\beta)} = \varphi_0^{(-\beta)} = U(-\beta^{-1})\varphi_0 = U^{-1}(\beta^{-1})\varphi_0$ and $\|(A_\beta^\dagger)^k \Psi_0^{(\beta)}\| \leq k!\, e^{2/\beta}$, $k \geq 0$. Hence

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Calling $N_\beta = B_\beta A_\beta$ and $\mathcal{N}_\beta = N_\beta^\dagger = A_\beta^\dagger B_\beta^\dagger$, since

$$N_\beta\, \varphi_n^{(\beta)} = n\, \varphi_n^{(\beta)}, \qquad \mathcal{N}_\beta \Psi_n^{(\beta)} = n\, \Psi_n^{(\beta)}, \tag{4}$$

these vectors above are biorthogonal and the following holds:

$$\left\langle \varphi_n^{(\beta)}, \Psi_m^{(\beta)} \right\rangle = \delta_{n,m}\, e^{-2/\beta^2}. \tag{5}$$

Both $\mathcal{F}_{\varphi}^{(\beta)} = \{\varphi_n^{(\beta)},\ n \geq 0\}$ and $\mathcal{F}_{\Psi}^{(\beta)} = \{\Psi_n^{(\beta)},\ n \geq 0\}$ are complete in $\mathcal{H}$: $< f, \varphi_n^{(\beta)} >= 0$ for all n iff $f = 0$. Hence Assumption 3 is satisfied.

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No: they are related to an orthonormal basis via the following self-adjoint, unbounded and invertible operator: $V_\beta = e^{(a+a^\dagger)/\beta}$, where $[a, a^\dagger] = \mathbb{1}$. More explicitly, we have $\varphi_k^{(\beta)} = e^{-1/\beta^2} V_\beta \varphi_k$. and $\Psi_k^{(\beta)} = e^{-1/\beta^2} V_\beta^{-1} \varphi_k$, where $\varphi_k = \frac{(a^\dagger)^k}{\sqrt{k!}} \varphi_0$, and $a\varphi_0 = 0$.

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Moreover, calling $h_\beta = \beta(a^\dagger a + \gamma_\beta \mathbb{1}) = h_\beta^\dagger$, we have

$$H_\beta V_\beta = V_\beta h_\beta : \tag{6}$$

Both $\mathcal{F}_{\varphi}^{(\beta)} = \{\varphi_n^{(\beta)},\ n \geq 0\}$ and $\mathcal{F}_{\Psi}^{(\beta)} = \{\Psi_n^{(\beta)},\ n \geq 0\}$ are complete in $\mathcal{H}$: $< f, \varphi_n^{(\beta)} >= 0$ for all n iff $f = 0$. Hence Assumption 3 is satisfied.

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No: they are related to an orthonormal basis via the following self-adjoint, unbounded and invertible operator: $V_\beta = e^{(a+a^\dagger)/\beta}$, where $[a, a^\dagger] = \mathbb{1}$. More explicitly, we have $\varphi_k^{(\beta)} = e^{-1/\beta^2} V_\beta \varphi_k$. and $\Psi_k^{(\beta)} = e^{-1/\beta^2} V_\beta^{-1} \varphi_k$, where $\varphi_k = \frac{(a^\dagger)^k}{\sqrt{k!}} \varphi_0$, and $a\varphi_0 = 0$.

Moreover, calling $h_\beta = \beta(a^\dagger a + \gamma_\beta \mathbb{1}) = h_\beta^\dagger$, we have

$$H_\beta V_\beta = V_\beta h_\beta : \tag{6}$$

V_β is an intertwining operator between h_β and H_β.

III.2. The Swanson hamiltonian

The non self-adjoint hamiltonian is

$$H_\theta = \frac{1}{2}\left(p^2 + x^2\right) - \frac{i}{2}\tan(2\theta)\left(p^2 - x^2\right),$$

$\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \setminus \{0\} =: I$. Introducing a and $a^\dagger$ we write

$$H_\theta = N + \frac{i}{2}\tan(2\theta)\left(a^2 + (a^\dagger)^2\right) + \frac{1}{2}\mathbb{1},$$

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then

$$H_\theta = \omega_\theta\left(B_\theta A_\theta + \frac{1}{2}\mathbb{1}\right), \qquad (7)$$

where $\omega_\theta = \frac{1}{\cos(2\theta)}$. We have $A_\theta^\dagger \neq B_\theta$ and $[A_\theta, B_\theta] = \mathbb{1}$.

$$A_\theta \varphi_0^{(\theta)} = 0 \Rightarrow$$

$$\varphi_0^{(\theta)}(x) = N_1 \exp\left\{-\frac{1}{2}\, e^{2i\theta}\, x^2\right\}, \qquad (8)$$

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We find:

$$\varphi_n^{(\theta)}(x) = \frac{1}{\sqrt{n!}}\, B_\theta^n\, \varphi_0^{(\theta)}(x) = \frac{N_1}{\sqrt{2^n\, n!}}\, H_n\left(e^{i\theta}x\right)\, \exp\left\{-\frac{1}{2}\, e^{2i\theta}\, x^2\right\},$$

$$\Psi_n^{(\theta)}(x) = \frac{1}{\sqrt{n!}}\, (A_\theta^\dagger)^n\, \Psi_0^{(\theta)}(x) = \frac{N_2}{\sqrt{2^n\, n!}}\, H_n\left(e^{-i\theta}x\right)\, \exp\left\{-\frac{1}{2}\, e^{-2i\theta}\, x^2\right\},$$

where $H_n(x)$ is the n-th Hermite polynomial.

Their norms are

$$\|\varphi_n^{(\theta)}\|^2 = |N_1|^2 \cos\left(\frac{\pi}{\cos(2\theta)}\right) P_n\left(\frac{1}{\cos(2\theta)}\right),$$

$$\|\Psi_n^{(\theta)}\|^2 = |N_2|^2 \cos\left(\frac{\pi}{\cos(2\theta)}\right) P_n\left(\frac{1}{\cos(2\theta)}\right),$$

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(i) complete in $\mathcal{L}^2(\mathbb{R})$;

(ii) Riesz bases.

Completeness [Kolmogorov and Fomin]: *if $\rho(x)$ is a Lebesgue-measurable function which is different from zero almost everywhere (a.e.) in $\mathbb{R}$ and if there exist two positive constants δ, C such that $|\rho(x)| \leq C\, e^{-\delta|x|}$ a.e. in $\mathbb{R}$, then the set $\{x^n\, \rho(x)\}$ is complete in $\mathcal{L}^2(\mathbb{R})$.*

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Riesz bases?: we introduce the unbounded, self-adjoint and invertible operator $T_\theta = e^{i\frac{\theta}{2}(a^2 - a^{\dagger 2})}$. Then

$$A_\theta = T_\theta a T_\theta^{-1}, \qquad B_\theta = T_\theta a^\dagger T_\theta^{-1}. \tag{10}$$

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T_θ is an IO: let $h_\theta = \omega_\theta \left(a^\dagger a + \frac{1}{2}\mathbb{1}\right) = h_\theta^\dagger$, then

$$H_\theta T_\theta = T_\theta h_\theta, \qquad T_\theta H_\theta^\dagger = h_\theta T_\theta, \tag{11}$$

and $\alpha \in \mathbb{C}$ exists such that

$$\varphi_n^{(\theta)} = \alpha\, T_\theta\, \varphi_n, \qquad \text{and} \qquad \Psi_n^{(\theta)} = \frac{1}{\alpha}\, T_\theta^{-1}\, \varphi_n \tag{12}$$

$\Rightarrow$ nor $\mathcal{F}_\varphi^{(\theta)}$ neither $\mathcal{F}_\Psi^{(\theta)}$ are Riesz bases: our pseudo-bosons are non-regular. Also, we deduce that $\eta_\varphi^{(\theta)} = |\alpha|^2\, T_\theta^2$ and $\eta_\Psi^{(\theta)} = |\alpha|^{-2}\, T_\theta^{-2}$. This is in agreement with the following (formal) computations:

$$\sum_{n=0}^{\infty} \left|\varphi_n^{(\theta)}\rangle\langle\,\Psi_n^{(\theta)}\right| = \alpha\, T_\theta \left(\sum_{n=0}^{\infty} |\varphi_n\rangle\langle\,\varphi_n|\right) \frac{1}{\alpha}\, T_\theta^{-1} = \mathbb{1},$$

$$\sum_{n=0}^{\infty} \left|\varphi_n^{(\theta)}\rangle\langle\,\varphi_n^{(\theta)}\right| = \alpha\, T_\theta \left(\sum_{n=0}^{\infty} |\varphi_n\rangle\langle\,\varphi_n|\right) (\alpha T_\theta)^\dagger = |\alpha|^2 T_\theta^2 = S_\varphi^{(\beta)},$$

as well as

$$\sum_{n=0}^{\infty} \left|\Psi_n^{(\theta)}\rangle\langle\,\Psi_n^{(\theta)}\right| = \frac{1}{\overline{\alpha}}\, T_\theta^{-1} \left(\sum_{n=0}^{\infty} |\varphi_n\rangle\langle\,\varphi_n|\right) \left(\frac{1}{\overline{\alpha}}\, T_\theta^{-1}\right)^\dagger =$$

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More (physically motivated) examples

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IV. Connections with bosons

We have considered the following question: *which is the relation between (regular) pseudo-bosons and ordinary bosons?* The answer is given by the following theorems [F. B., J. Phys. A, **44**, 015205 (2011)]:

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Theorem 1 *Let a and b be such that $[a, b] = \mathbb{1}$, and for which Assumptions 1-4 are satisfied. Then an unbounded, densely defined, operator c on $\mathcal{H}$ exists, and a positive bounded operator T with bounded inverse T^{-1}, such that $[c, c^\dagger] = \mathbb{1}$. Moreover*

$$a = T\,c\,T^{-1}, \qquad b = T\,c^\dagger T^{-1}. \tag{1}$$

Viceversa, given an unbounded, densely defined, operator c on $\mathcal{H}$ satisfying $[c, c^\dagger] = \mathbb{1}$ and a positive bounded operator T with bounded inverse T^{-1}, two operators a and b can be introduced for which $[a, b] = \mathbb{1}$, and for which equations (1) and Assumptions 1-4 are satisfied.

Theorem 2 *Let a and b be such that $[a, b] = \mathbb{1}$, and Assumptions 1-3 (but not 4) hold true. Then two unbounded, densely defined, operators c and R on $\mathcal{H}$ exist, such that $[c, c^\dagger] = \mathbb{1}$ and R is positive, self adjoint and with unbounded inverse R^{-1}. Moreover*

$$a = RcR^{-1}, \qquad b = Rc^\dagger R^{-1}, \tag{2}$$

and, introducing $\hat{\varphi}_n = \frac{c^{\dagger n}}{\sqrt{n!}}\hat{\varphi}_0$, $c\varphi_0 = 0$, then $\hat{\varphi}_n \in D(R) \cap D(R^{-1})$, for all $n \geq 0$, and the sets $\{R\hat{\varphi}_n\}$ and $\{R^{-1}\hat{\varphi}_n\}$ are biorthogonal bases of $\mathcal{H}$.

Viceversa, let us consider two unbounded, densely defined, operators c and R on $\mathcal{H}$ satisfying $[c, c^\dagger] = \mathbb{1}$ with R positive, self-adjoint with unbounded inverse R^{-1}. Suppose that, introduced $\hat{\varphi}_n$ as above, $\hat{\varphi}_n \in D(R) \cap D(R^{-1})$, for all $n \geq 0$, and that the sets $\{R\hat{\varphi}_n\}$ and $\{R^{-1}\hat{\varphi}_n\}$ are biorthogonal bases of $\mathcal{H}$. Then two operators a and b can be introduced for which $[a, b] = \mathbb{1}$, and for which equations (2) and Assumptions 1-3 (but not 4) are satisfied.

V. Non-linear pseudo-bosons

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We use an idea imported from non-linear coherent states:

$$|z>= e^{-|z|^2/2} \sum_{k=0}^{\infty} \frac{z^n}{\sqrt{n!}} \Phi_n$$

becomes

$$\Xi(z) := N(|z|^2)^{-1/2} \sum_{k=0}^{\infty} \frac{z^n}{\sqrt{\epsilon_n!}} \Phi_n,$$

where $\epsilon_n! = \epsilon_1 \cdots \epsilon_n$, with $\epsilon_0! = 1$ and $N(|z|^2)$ a proper normalization (inside a certain domain of convergence).

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Let a and b be operators on $\mathcal{H}$ and $\{\epsilon_n\}$ such that $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \cdots$. Then [F. B., J. Math. Phys., **52**, 063521, (2011)]..

..the triple $(a, b, \{\epsilon_n\})$ is a family of non-linear regular pseudo-bosons (NLRPB) if:

- **p1.** a non zero vector Φ_0 exists in $\mathcal{H}$ such that $a\,\Phi_0 = 0$ and $\Phi_0 \in D^\infty(b)$.

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- **p3.** Calling

$$\Phi_n := \frac{1}{\sqrt{\epsilon_n!}}\, b^n\,\Phi_0, \qquad \eta_n := \frac{1}{\sqrt{\epsilon_n!}}\, {a^\dagger}^n\,\eta_0,$$

we have, for all $n \geq 0$,

$$a\,\Phi_n = \sqrt{\epsilon_n}\,\Phi_{n-1}, \qquad b^\dagger\eta_n = \sqrt{\epsilon_n}\,\eta_{n-1}.$$

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- **p4.** $\mathcal{F}_\Phi = \{\Phi_n,\, n \geq 0\}$ and $\mathcal{F}_\eta = \{\eta_n,\, n \geq 0\}$ are bases of $\mathcal{H}$.
- **p5.** $\mathcal{F}_\Phi$ and $\mathcal{F}_\eta$ are Riesz bases of $\mathcal{H}$.

Let us introduce the following (not self-adjoint) operators:

$$M = ba, \qquad \mathfrak{M} = M^\dagger = a^\dagger b^\dagger. \tag{1}$$

Then we can check that $\Phi_n \in D(M) \cap D(b)$, $\eta_n \in D(\mathfrak{M}) \cap D(a^\dagger)$, and that

$$b\,\Phi_n = \sqrt{\epsilon_{n+1}}\,\Phi_{n+1}, \qquad a^\dagger \eta_n = \sqrt{\epsilon_{n+1}}\,\eta_{n+1}, \tag{2}$$

as well as

$$M\Phi_n = \epsilon_n \Phi_n, \qquad \mathfrak{M}\eta_n = \epsilon_n \eta_n, \tag{3}$$

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Hence, if $\langle \Phi_0, \eta_0 \rangle = 1$,

$$\langle \Phi_n, \eta_m \rangle = \delta_{n,m}, \tag{4}$$

Moreover

$$\sum_n |\Phi_n >< \eta_n| = \sum_n |\eta_n >< \Phi_n| = \mathbb{1}, \qquad (5)$$

while **p5** implies that $S_\Phi := \sum_n |\Phi_n >< \Phi_n|$ and $S_\eta := \sum_n |\eta_n >< \eta_n|$ are positive, bounded, invertible and that $S_\Phi = S_\eta^{-1}$.

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The new fact is that the operators a and b do not, in general, satisfy any simple commutation rule. Indeed, we can check that, for all $n \geq 0$,

$$[a, b]\Phi_n = (\epsilon_{n+1} - \epsilon_n)\, \Phi_n, \qquad (6)$$

which is different from $[a, b] = \mathbb{1}$, except if $\epsilon_n = n$. We end this overview mentioning also that M and $\mathfrak{M}$ are connected by an intertwining operator:

$$M S_\Phi = S_\Phi \mathfrak{M}$$

VI. Relation with cryptohermiticity

With M. Znojil we have analyzed the connections between NLRPB and CH in JPA, 2011. The starting point is the following

Definition 3 *Let us consider two operators H and Θ acting on the Hilbert space $\mathcal{H}$, with Θ positive and invertible. Let us call $H^\dagger$ the adjoint of H in $\mathcal{H}$ with respect to its scalar product and $H^\ddagger = \Theta^{-1} H^\dagger \Theta$, when this exists. We will say that H is cryptohermitian with respect to Θ (CHwrtΘ) if $H = H^\ddagger$.*

We will restrict here to Θ and Θ^{-1} bounded. The operators $\Theta^{\pm 1/2}$ are well defined. Hence we can introduce an operator $h := \Theta^{1/2}\, H\, \Theta^{-1/2}$. It is easy to check that $h = h^\dagger$. Hence the following definition appears natural:

Definition 4 *Assume that H is CHwrtΘ, for H and Θ as above. H is well behaved wrt Θ if h has only discrete eigenvalues ϵ_n, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, with eigenvectors e_n: $he_n = \epsilon_n e_n$, $n \in \mathbb{N}_0$, and $\mathcal{E} = \{e_n\}$ is a basis of $\mathcal{H}$.*

Useful technical assumptions:

1. the multiplicity of each eigenvalue ϵ_n is one.
2. We assume $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \ldots$.

Definition 4 *Assume that H is CHwrtΘ, for H and Θ as above. H is well behaved wrt Θ if h has only discrete eigenvalues ϵ_n, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, with eigenvectors e_n: $he_n = \epsilon_n e_n$, $n \in \mathbb{N}_0$, and $\mathcal{E} = \{e_n\}$ is a basis of $\mathcal{H}$.*

Useful technical assumptions:

1. the multiplicity of each eigenvalue ϵ_n is one.
2. We assume $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \ldots$.

Theorem 5 *Let H be well behaved wrt Θ, where $\Theta, \Theta^{-1} \in B(\mathcal{H})$, and $\Theta = \Theta^\dagger$. Then it is possible to introduce two operators a and b on $\mathcal{H}$, and a sequence of real numbers $\{\epsilon_n,\ n \in \mathbb{N}_0\}$, such that the triple $(a, b, \{\epsilon_n\})$ is a family of NLRPB.*
Vice versa, if $(a, b, \{\epsilon_n\})$ is a family of NLRPB, two operators can be introduced, H and Θ, such that $\Theta, \Theta^{-1} \in B(\mathcal{H})$ and $\Theta = \Theta^\dagger$, and H is well behaved wrt Θ.

Consequences:

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1. Formally we have

$$a = \sum_{n=0}^{\infty} \sqrt{\epsilon_n} |\Phi_{n-1} > < \eta_n|, \quad b = \sum_{n=0}^{\infty} \sqrt{\epsilon_{n+1}} |\Phi_{n+1} > < \eta_n|,$$

as well as

$$h = \sum_{n=0}^{\infty} \epsilon_n |e_n > < e_n|$$

,

$$H = \sum_{n=0}^{\infty} \epsilon_n |\Phi_n > < \eta_n|$$

and

$$H^\dagger = \sum_{n=0}^{\infty} \epsilon_n |\eta_n > < \Phi_n|.$$

In particular h, H and $H^\dagger$ are isospectrals.

2. Even if h is not required to be factorizable, because of our construction it turns out that it can be written as $h = b_\Theta a_\Theta$, where $a_\Theta = \Theta^{1/2} a\, \Theta^{-1/2}$ and $b_\Theta = \Theta^{1/2} b\, \Theta^{-1/2}$. Incidentally, in general $[a_\Theta, b_\Theta] = \Theta^{1/2}[a, b]\, \Theta^{-1/2} \neq [a, b]$, but if $\left[[a, b], \Theta^{1/2}\right] = 0$, which is the case for pseudo-bosons. Therefore, at least at a formal level, our construction shows that the hamiltonian h can be written in a factorized form.

VII. Pseudo-fermions

[F.B., J. Phys. A, 2012]

The CAR are replaced here by the following rules:

$$\{a, b\} = \mathbb{1}, \quad \{a, a\} = 0, \quad \{b, b\} = 0, \qquad (1)$$

where the relevant situation is when $b \neq a^\dagger$. Compared with Assumptions 1-4 for PB, the only assumptions we might need to require now are the following

- **p1.** a non zero vector φ_0 exists in $\mathcal{H}$ such that $a\,\varphi_0 = 0$.
- **p2.** a non zero vector Ψ_0 exists in $\mathcal{H}$ such that $b^\dagger\,\Psi_0 = 0$.

However, even these two requirements are automatically satisfied, as a consequence of (1):

In fact, in $\mathcal{H}$, it is easy to check that the only non-trivial possible choices of a and b satisfying (1) are the following:

$$a(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b(1) = \begin{pmatrix} \beta & -\beta^2 \\ 1 & -\beta \end{pmatrix},$$

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$$a(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b(1) = \begin{pmatrix} \beta & -\beta^2 \\ 1 & -\beta \end{pmatrix},$$

$$a(2) = \begin{pmatrix} \alpha & 1 \\ -\alpha^2 & -\alpha \end{pmatrix}, \quad b(2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with non zero α and β,

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with non zero α and β,
or, maybe more interestingly,

$$a(3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ -\alpha_{11}^2/\alpha_{12} & -\alpha_{11} \end{pmatrix}, \quad b(3) = \begin{pmatrix} \beta_{11} & \beta_{12} \\ -\beta_{11}^2/\beta_{12} & -\beta_{11} \end{pmatrix},$$

with $2\alpha_{11}\beta_{11} - \frac{\alpha_{11}^2\beta_{12}}{\alpha_{12}} - \frac{\beta_{11}^2\alpha_{12}}{\beta_{12}} = 1$. For all these choices, it is easy to show that the two non zero vectors φ_0 and Ψ_0 of **p1** and **p2** do exist. This is not surprising, since $\det(a) = \det(b^\dagger) = 0$.

For instance, if we take $\alpha_{11} = \frac{1}{3}$, $\beta_{11} = \frac{2}{3}$, and $\alpha_{12} = -\beta_{12} = -i$, we find:

$$a(3) = \begin{pmatrix} 1/3 & -i \\ -i/9 & -1/3 \end{pmatrix}, \; b(3) = \begin{pmatrix} 2/3 & i \\ 4i/9 & -2/3 \end{pmatrix},$$

$$\varphi_0 = \alpha \begin{pmatrix} 1 \\ -i/3 \end{pmatrix}, \; \Psi_0 = \beta \begin{pmatrix} 1 \\ -3i/2 \end{pmatrix}.$$

It is not difficult to relate α and β in such a way $\langle \varphi_0, \Psi_0 \rangle = 1$.

It is now possible to recover similar results as those for PB. In particular, we introduce

$$\varphi_1 := b\varphi_0, \quad \Psi_1 = a^\dagger \Psi_0, \tag{2}$$

as well as the non self-adjoint operators

$$N = ba, \quad \mathcal{N} = N^\dagger = a^\dagger b^\dagger. \tag{3}$$

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We further introduce S_φ and S_Ψ:

$$S_\varphi f = \sum_{n=0}^{1} \langle \varphi_n, f \rangle \, \varphi_n, \quad S_\Psi f = \sum_{n=0}^{1} \langle \Psi_n, f \rangle \, \Psi_n, \tag{4}$$

$f \in \mathcal{H}$. Hence we get:

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$f \in \mathcal{H}$. Hence we get:

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$$a\varphi_1 = \varphi_0, \quad b^\dagger \Psi_1 = \Psi_0. \tag{5}$$

2.
$$N\varphi_n = n\varphi_n, \quad \mathcal{N}\Psi_n = n\Psi_n, \tag{6}$$

3. If $\langle \varphi_0, \Psi_0 \rangle = 1$, then

$$\langle \varphi_k, \Psi_n \rangle = \delta_{k,n}, \tag{7}$$

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for $k, n = 0, 1$.

4. S_φ and S_Ψ are bounded, strictly positive, self-adjoint, and invertible. They satisfy

$$\|S_\varphi\| \leq \|\varphi_0\|^2 + \|\varphi_1\|^2, \quad \|S_\Psi\| \leq \|\Psi_0\|^2 + \|\Psi_1\|^2,$$

$$S_\varphi \Psi_n = \varphi_n, \qquad S_\Psi \varphi_n = \Psi_n, \tag{8}$$

for $n = 0, 1$, as well as $S_\varphi = S_\Psi^{-1}$ and the following intertwining relations

$$S_\Psi N = \mathcal{N} S_\Psi, \qquad S_\varphi \mathcal{N} = N S_\varphi. \tag{9}$$

Then:

(i) N and $\mathcal{N}$ behave as fermionic number operators, having eigenvalues 0 and 1;

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(vi) the very well-behaved operators S_φ and S_Ψ maps $\mathcal{F}_\varphi$ in $\mathcal{F}_\Psi$ and viceversa;

(vii) S_φ and S_Ψ intertwine between operators which are not self-adjoint, in the very same way as they do for PB.

The Assumptions 1-4 are automatically satisfied: we get Riesz bases for free, and we don't need to impose conditions on the domains of operators. Also:

Theorem 6 *Let c and $T = T^\dagger$ be two operators on $\mathcal{H}$ such that $\{c, c^\dagger\} = \mathbb{1}$, $c^2 = 0$, and $T > 0$. Then, defining*

$$a = T\, c\, T^{-1}, \quad b = T\, c^\dagger\, T^{-1}, \tag{10}$$

these operators satisfy (1).
Viceversa, given two operators a and b acting on $\mathcal{H}$, satisfying (1), it is possible to define two operators, c and T, such that $\{c, c^\dagger\} = \mathbb{1}$, $c^2 = 0$, $T = T^\dagger$ is strictly positive, and (10) holds.

VIII. Application to decay

The starting point is the Schrödinger equation

$$i\dot{\Psi}(t) = H_{eff}\Psi(t), \text{ with } H_{eff} = \frac{1}{2}\begin{pmatrix} -i\gamma_a & v \\ \overline{v} & -i\gamma_b \end{pmatrix},$$

where $\gamma_a, \gamma_b > 0$ and $v \in \mathbb{C}$, [Ben-Aryeh etc., JPA, 2004; Trifonov etc., JPA, 2007].

VIII.0.1. Schrödinger representation

Putting $\Phi(t) = e^{\Gamma t}\Psi(t)$, $\Gamma = \frac{1}{2}(\gamma_a + \gamma_b)$, we get $i\dot{\Phi}(t) = H\Phi(t)$, where

$$H = i\Gamma\mathbb{1}_2 + H_{eff} = \begin{pmatrix} -i\gamma & v \\ \overline{v} & i\gamma \end{pmatrix}, \quad \Phi(t) = \begin{pmatrix} \Phi_0(t) \\ \Phi_1(t) \end{pmatrix}.$$

Here $\gamma = \frac{1}{2}(\gamma_a - \gamma_b)$. Calling $\Omega := |v|^2 - \gamma^2$ we find

$$\begin{cases} \ddot{\Phi}_0(t) = -\Omega\,\Phi_0(t), \\ \ddot{\Phi}_1(t) = -\Omega\,\Phi_1(t). \end{cases}$$

$\Omega = 0$: the functions $\Phi_0(t)$ and $\Phi_1(t)$ are linear in t, so that

$$\Psi(t) = e^{-\Gamma t} \begin{pmatrix} \Phi_0(t) \\ \Phi_1(t) \end{pmatrix} = \begin{pmatrix} e^{-(\gamma_a+\gamma_b)\frac{t}{2}}(A_0 + B_0\, t) \\ e^{-(\gamma_a+\gamma_b)\frac{t}{2}}(A_1 + B_1\, t) \end{pmatrix}.$$

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$\Omega > 0$. In this case the solution can be written as

$$\Psi(t) = e^{-(\gamma_a+\gamma_b)\frac{t}{2}} \begin{pmatrix} A_0 \cos(\sqrt{\Omega}\, t) + B_0 \sin(\sqrt{\Omega}\, t) \\ A_1 \cos(\sqrt{\Omega}\, t) + B_1 \sin(\sqrt{\Omega}\, t) \end{pmatrix},$$

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$\Omega < 0$. In this case the solution can be written as

$$\Psi(t) = e^{-(\gamma_a+\gamma_b)\frac{t}{2}} \begin{pmatrix} A_0 \exp(\sqrt{|\Omega|}\, t) + B_0 \exp(-\sqrt{|\Omega|}\, t) \\ A_1 \exp(\sqrt{|\Omega|}\, t) + B_1 \exp(-\sqrt{|\Omega|}\, t). \end{pmatrix}.$$

Here A_0, A_1, B_0 and B_1 are fixed by the initial conditions.

In all cases, when $t \to \infty$, even if in general $\|\Phi(t)\| \nrightarrow 0$, we find that

$$\|\Psi(t)\| \to 0.$$

VIII.0.2. Heisenberg representation

The eigenvalues of H can be written as $\lambda_\pm := \pm\sqrt{\Omega}$, and the eigenstates are

$$\eta_+ = \begin{pmatrix} \frac{1}{\bar{v}}\left(-i\gamma + \sqrt{\Omega}\right) \\ 1 \end{pmatrix}, \; \eta_- = \begin{pmatrix} -\frac{1}{\bar{v}}\left(i\gamma + \sqrt{\Omega}\right) \\ 1 \end{pmatrix}.$$

Notice that $< \eta_+, \eta_- >= \frac{2\gamma}{|v|^2}\left(\gamma - i\sqrt{\Omega}\right)$, which is zero only if $\gamma = 0$ ($H = H^\dagger$) or if $\gamma = i\sqrt{\Omega}$ ($H = -H^\dagger$). Also, going back to H_{eff}

$$H_{eff}\,\eta_\pm = E_\pm\,\eta_\pm, \qquad E_\pm = -\frac{i}{2}(\gamma_a + \gamma_b) \pm \sqrt{\Omega}.$$

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It is possible now to introduce two operators a and b, such that $\{a, b\} = \mathbb{1}$, $a^2 = b^2 = 0$, and

$$H = \Omega\left(b\,a - \frac{1}{2}\mathbb{1}\right) = \Omega\left(N - \frac{1}{2}\mathbb{1}\right),$$

where $N = b\,a$.

To recover the same damping we have found in Schrödinger representation, it is natural to consider the time evolution of the number operator N:

$$N_{eff}(t) = e^{iH_{eff}^{\dagger}\,t}\,N\,e^{-iH_{eff}\,t},$$

which turns out to be

$$N_{eff}(t) = e^{-2\Gamma\,t}\left(Ne^{-i\Omega t} + \mathcal{N}\,N(1 - e^{-i\Omega t})\right).$$

Then, if we estimate the norm of $N_{eff}(t)$, it is trivial to deduce that

$$\|N_{eff}(t)\| \leq 3e^{-2\Gamma t},$$

which goes to zero when t diverges. Hence, as expected, we recover damping also in Heisenberg picture.

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Remark:– Larger dimensional examples can also be constructed, see FB, J. Phys. A, submitted.

IX. What more?

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