First, there’s the room you can see through the glass – that’s just the same as our drawing-room, only the things go the other way.
1. QM with non-Hermitian operators
(just some conceptual remarks)

2. $\mathcal{PT}$-symmetry
(what is known and my point of view)

3. physical $\mathcal{PT}$-symmetric models in QM
(non-self-adjoint Robin boundary conditions)

4. imaginary cubic oscillator
(about the non-existence of the metric operator)

5. Conclusions
¿ QM with non-Hermitian operators?

\[ H^* = H \]

\[ H^{PT} = H \]

*Imaginary Numbers* by Yves Tanguy, 1954
(Museo Thyssen-Bornemisza, Madrid)
Example 1. evolution operator $U(t) = \exp(-itH)$:

\[
\begin{align*}
    i\dot{U}(t) &= HU(t) \\
    U(0) &= I
\end{align*}
\]
Insignificant non-Hermiticity

Example 1. evolution operator \( U(t) = \exp(-itH) : \) \[
\begin{align*}
i \dot{U}(t) &= H U(t) \\
U(0) &= I
\end{align*}
\]

Example 2. resolvent operator \( R(z) = (H - z)^{-1}, \quad z \in \mathbb{C} \)
Example 1. evolution operator $U(t) = \exp(-itH)$:

\[
\begin{aligned}
i \dot{U}(t) &= HU(t) \\
U(0) &= I
\end{aligned}
\]

Example 2. resolvent operator $R(z) = (H - z)^{-1}$, $z \in \mathbb{C}$

**Theorem (spectral theorem).**

Let $H = H^*$. Then

\[
f(H) = \int_{\sigma(H)} f(\lambda) \, dE_H(\lambda)
\]

for any complex-valued continuous function $f$. 
Technical non-Hermiticity
Example 1. complex scaling \( H_\theta := S_\theta(-\Delta + V)S_\theta^{-1} \), \( (S_\theta \psi)(x) := e^{\theta/2} \psi(e^{\theta} x) \)

\[ \theta = 0 \quad \text{and} \quad \Im \theta > 0 \]

[Aguilar/Balslev, Combes 1971], [Simon 1972], [Van Winter 1974], . . .
Example 1. complex scaling \( H_\theta := S_\theta(-\Delta + V)S_\theta^{-1}, \quad (S_\theta \psi)(x) := e^{\theta/2} \psi(e^\theta x) \)

\[ \theta = 0 \quad \text{and} \quad \Im \theta > 0 \]

[Example 2. adiabatic transition probability for \( H(t) := \tilde{\gamma}(t/\tau) \cdot \tilde{\sigma}, \quad \tau \to \infty \)]

[Aguilar/Balslev, Combes 1971], [Simon 1972], [Van Winter 1974], . . .

[Berry 1990], [Joye, Kunz, Pfister 1991], [Jakšić, Segert 1993], . . .
Technical non-Hermiticity

Example 1. complex scaling  $H_\theta := S_\theta (-\Delta + V) S_\theta^{-1}, \quad (S_\theta \psi)(x) := e^{\theta/2} \psi(e^\theta x)$

$\theta = 0$

$\Im \theta > 0$

[Aguilar/Balslev, Combes 1971], [Simon 1972], [Van Winter 1974], . . .

Example 2. adiabatic transition probability for  $H(t) := \vec{\gamma}(t/\tau) \cdot \vec{\sigma}, \quad \tau \to \infty$

[Berry 1990], [Joye, Kunz, Pfister 1991], [Jakšić, Segert 1993], . . .

Example 3. Regge theory  $H_l := -\frac{d^2}{dr^2} + \frac{l(l + 1)}{r^2} + V(r), \quad l \in \mathbb{C}$

[Regge 1957], [Connor 1990], [Sokolovski 2011], . . .
Approximate non-Hermiticity

open systems

Example 1. radioactive decay

Example 2. dissipative Schrödinger operators in semiconductor physics

Baro, Behrndt, Kaiser, Neidhardt, Rehberg, . . .

Example 3. repeated interaction quantum systems

Bruneau, Joye, Merkli, Pillet, . . .
¿ Fundamental non-Hermiticity? 

i.e. non-Hermitian observables, 
without violating physical axioms of QM
¿ Fundamental non-Hermiticity? 

i.e. non-Hermitian observables, 
without violating physical axioms of QM

¡ no!

**Theorem (Stone’s theorem).**
Unitary groups on a *Hilbert space* are generated by *self-adjoint* operators.
¿ Fundamental non-Hermiticity? 

i.e. non-Hermitian observables, 
without violating physical axioms of QM

¡ no! 

**Theorem (Stone’s theorem).**
Unitary groups on a *Hilbert space* are generated by self-adjoint operators.

¡ yes? 

by changing the Hilbert space, preserving a *similarity* to self-adjoint operators
Non-Hermitian Hamiltonians with real spectra

\[-\Delta + V \text{ in } L^2(\mathbb{R})\]

\[V(x) = x^2 + ix^3\]

[Caliceti, Graffi, Maioli 1980]

\[V(x) = x^2 (ix)^\varepsilon\]

[Bessis, Zinn-Justin]
[Bender, Boettcher 1998]
[Dorey, Dunning, Tateo 2001]
[Shin 2002]
[Azizov, Kuzhel, Günther, Trunk 2010]

\[V(x) = \begin{cases} 
    i \text{sgn}(x) & \text{if } x \in (-L, L) \\
    \infty & \text{elsewhere}
\end{cases}\]

[Znojil 2001]

¿ What is behind the reality of the spectrum?
\( \mathcal{PT} - \text{symmetry} \)

\[
[H, \mathcal{PT}] = 0
\]

\[
(\mathcal{P}\psi)(x) := \psi(-x)
\]

\[
(\mathcal{I}\psi)(x) := \overline{\psi(x)}
\]

We have in mind \( H = -\Delta + V \) on \( L^2(\mathbb{R}^d) \) with \( V(-x) = V(x) \).
\[ [H, \mathcal{PT}] = 0 \]

\[
(\mathcal{P}\psi)(x) := \psi(-x) \\
(\mathcal{I}\psi)(x) := \bar{\psi}(x)
\]

We have in mind \( H = -\Delta + V \) on \( L^2(\mathbb{R}^d) \) with \( \overline{V(-x)} = V(x) \).

\( \mathcal{PT} \) is an antilinear symmetry \( \implies \) in general only: \( \lambda \in \sigma(H) \iff \bar{\lambda} \in \sigma(H) \)
\( \mathcal{PT} \)-symmetry

\[ [H, \mathcal{PT}] = 0 \]

\[(\mathcal{P}\psi)(x) := \psi(-x)\]
\[(\mathcal{J}\psi)(x) := \overline{\psi(x)}\]

We have in mind \( H = -\Delta + V \) on \( L^2(\mathbb{R}^d) \) with \( V(-x) = V(x) \).

\( \mathcal{PT} \) is an antilinear symmetry \( \implies \) in general only: \( \lambda \in \sigma(H) \iff \overline{\lambda} \in \sigma(H) \)

unbroken \( \mathcal{PT} \)-symmetry \( \iff \) \( H \) and \( \mathcal{PT} \) have the same eigenstates \( \iff \sigma(H) \subseteq \mathbb{R} \)

Here we assume that \( H \) has purely discrete spectrum.
**$\mathcal{PT}$-symmetry**

\[
[H, \mathcal{PT}] = 0
\]

\[
(\mathcal{P}\psi)(x) := \psi(-x)
\]

\[
(\mathcal{J}\psi)(x) := \overline{\psi(x)}
\]

We have in mind $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ with $V(-x) = V(x)$.

$\mathcal{PT}$ is an **antilinear** symmetry $\implies$ in general only: $\lambda \in \sigma(H) \iff \overline{\lambda} \in \sigma(H)$

unbroken $\mathcal{PT}$-symmetry $:\iff H$ and $\mathcal{PT}$ have the same eigenstates $\iff \sigma(H) \subset \mathbb{R}$

Here we assume that $H$ has purely discrete spectrum.

**perturbation-theory insight**

\[
H_0 + \varepsilon V =: H \leadsto \text{spectrum moves at most by } \varepsilon \|V\|
\]

self-adjoint $\quad \mathcal{PT}$-symmetric $\quad \iff$ simple eigenvalues remain real for small $\varepsilon$
\[ [H, \mathcal{PT}] = 0 \]

\[(\mathcal{P}\psi)(x) := \psi(-x) \]
\[(\mathcal{I}\psi)(x) := \overline{\psi(x)} \]

We have in mind \( H = -\Delta + V \) on \( L^2(\mathbb{R}^d) \) with \( V(-x) = V(x) \).

\( \mathcal{PT} \) is an antilinear symmetry \( \implies \) in general only: \( \lambda \in \sigma(H) \iff \bar{\lambda} \in \sigma(H) \)

unbroken \( \mathcal{PT} \)-symmetry :\(\iff\) \( H \) and \( \mathcal{PT} \) have the same eigenstates \( \iff \sigma(H) \subset \mathbb{R} \)

Here we assume that \( H \) has purely discrete spectrum.

**perturbation-theory insight**

\[ H_0 + \varepsilon V =: H \sim \text{spectrum moves at most by } \varepsilon \|V\| \]

self-adjoint \( \mathcal{PT} \)-symmetric \( \iff \) simple eigenvalues remain real for small \( \varepsilon \)

Moreover, let the eigenstates of \( H \) form a Riesz basis. \( H\psi_n = E_n\psi_n, \ H^*\phi_n = E_n\phi_n \)
\[ [H, \mathcal{PT}] = 0 \]

\[
(\mathcal{P}\psi)(x) := \psi(-x) \\
(\mathcal{J}\psi)(x) := \overline{\psi(x)}
\]

We have in mind \( H = -\Delta + V \) on \( L^2(\mathbb{R}^d) \) with \( V(-x) = V(x) \).

\( \mathcal{PT} \) is an antilinear symmetry \( \implies \) in general only: \( \lambda \in \sigma(H) \iff \bar{\lambda} \in \sigma(H) \)

unbroken \( \mathcal{PT} \)-symmetry \( \iff \) \( H \) and \( \mathcal{PT} \) have the same eigenstates \( \iff \sigma(H) \subset \mathbb{R} \)

Here we assume that \( H \) has purely discrete spectrum.

**perturbation-theory insight**

\[
H_0 + \varepsilon V =: H \sim \text{spectral moves at most by } \varepsilon \| V \|
\]

self-adjoint \( \mathcal{PT} \)-symmetric \( \iff^* \) simple eigenvalues remain real for small \( \varepsilon \)

Moreover, let the eigenstates of \( H \) form a Riesz basis. \( H\psi_n = E_n\psi_n, \ H^*\phi_n = E_n\phi_n \)

\( \iff \) \( H^* = \Theta H \Theta^{-1} \)

where \( \Theta := \sum_n \phi_n \langle \phi_n, \cdot \rangle \) is positive, bounded, boundedly invertible

\( \iff \) \( H \) is self-adjoint in \( (L^2, \langle \cdot, \Theta \cdot \rangle) \), \( i.e. \) \( \Theta^{1/2} H \Theta^{-1/2} \) is self-adjoint in \( (L^2, \langle \cdot, \cdot \rangle) \) metric
\[ [H, \mathcal{PT}] = 0 \]
\[
(\mathcal{P}\psi)(x) := \psi(-x) \\
(\mathcal{T}\psi)(x) := \overline{\psi(x)}
\]

We have in mind \( H = -\Delta + V \) on \( L^2(\mathbb{R}^d) \) with \( \overline{V(-x)} = V(x) \).

\( \mathcal{PT} \) is an antilinear symmetry \( \implies \) in general only: \( \lambda \in \sigma(H) \Leftrightarrow \bar{\lambda} \in \sigma(H) \)

unbroken \( \mathcal{PT} \)-symmetry \( \iff \) \( H \) and \( \mathcal{PT} \) have the same eigenstates \( \iff \sigma(H) \subset \mathbb{R} \)

Here we assume that \( H \) has purely discrete spectrum.

perturbation-theory insight

\( H_0 + \varepsilon V =: H \) \( \sim \) spectrum moves at most by \( \varepsilon \|V\| \)

self-adjoint \( \quad \mathcal{PT} \)-symmetric \( \quad \Rightarrow \) \( \ast \) simple eigenvalues remain real for small \( \varepsilon \)

Moreover, let the eigenstates of \( H \) form a Riesz basis. \( H\psi_n = E_n\psi_n, H^*\phi_n = E_n\phi_n \)

\( \Rightarrow \quad H^* = \Theta H \Theta^{-1} \) where \( \Theta := \sum_n \phi_n \langle \phi_n, \cdot \rangle \) is positive, bounded, boundedly invertible

\( \Rightarrow \quad H \) is self-adjoint in \( (L^2, \langle \cdot, \Theta \cdot \rangle) \), \( \text{i.e.} \ \Theta^{1/2}H\Theta^{-1/2} \) is self-adjoint in \( (L^2, \langle \cdot, \cdot \rangle) \) metric

Albeverio-Fei-Kurasov, Bender-Brody-Jones, Caliceti-Graffi-Sjöstrand, Fring, Graefe-Schubert, Kretschmer-Szymanowski, Langer-Tretter, Mostafazadeh, Scholtz-Geyer-Hahne, Znojil, \ldots
Speculations about “unbounded metric”

[Kretschmer, Szymanowski 2004], [Mostafazadeh 2012], [Bender, Kuzhel 2012]
Speculations about “unbounded metric”

[H. Kretschmer, Szymanowski 2004], [Mostafazadeh 2012], [Bender, Kuzhel 2012]

\[ H := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ on } \mathbb{C}^2 \]

satisfies \( H^* \Theta = \Theta H \)

with \( \Theta := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \)
Speculations about “unbounded metric”

[Kretschmer, Szymanowski 2004], [Mostafazadeh 2012], [Bender, Kuzhel 2012]

\[ H := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ on } \mathbb{C}^2 \]

satisfies \( H^* \Theta = \Theta H \) with \( \Theta := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \)

↓ Mostafazadeh’s construction

\[ h = 1 \text{ on } \mathbb{C} \]
Speculations about “unbounded metric”

[Kretschmer, Szymanowski 2004], [Mostafazadeh 2012], [Bender, Kuzhel 2012]

\[
H := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ on } \mathbb{C}^2 \quad \text{satisfies} \quad H^* \Theta = \Theta H \quad \text{with} \quad \Theta := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

Mostafazadeh’s construction

\[h = 1 \quad \text{on } \mathbb{C}\]

In \(\infty\)-dimensional spaces:

\[\exists \text{ similar examples with } \Theta > 0 \text{ invertible but } \Theta^{-1} \text{ unbounded} \]

\[\exists \text{ possible } \langle \phi_n, \psi_n \rangle \neq 0 \text{ for all } n \text{ but } \langle \phi_n, \psi_n \rangle \xrightarrow{n \to \infty} 0 \]

\[\exists \text{ unbounded } \Theta \text{ or } \Theta^{-1} \text{ always exist} \]
Speculations about “unbounded metric”

[Kretschmer, Szymanowski 2004], [Mostafazadeh 2012], [Bender, Kuzhel 2012]

\[ H := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ on } \mathbb{C}^2 \text{ satisfies } H^* \Theta = \Theta H \text{ with } \Theta := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

Mostafazadeh’s construction

\[ h = 1 \text{ on } \mathbb{C} \]

In \( \infty \)-dimensional spaces: i similar examples with \( \Theta > 0 \) invertible but \( \Theta^{-1} \) unbounded!

i possible \( \langle \phi_n, \psi_n \rangle \neq 0 \) for all \( n \) but \( \langle \phi_n, \psi_n \rangle \xrightarrow[n \to \infty]{} 0 \)!

i unbounded \( \Theta \) or \( \Theta^{-1} \) always exist!

Moreover: i physically relevant quantities are not preserved!

(continuous spectrum, pseudospectrum)

i spectral instabilities!
Speculations about “unbounded metric”

[Kretschmer, Szymanowski 2004], [Mostafazadeh 2012], [Bender, Kuzhel 2012]

\[ H := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ on } \mathbb{C}^2 \]
satisfies \[ H^* \Theta = \Theta H \]
with \[ \Theta := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

Mostafazadeh’s construction

\[ h = 1 \text{ on } \mathbb{C} \]

In \( \infty \)-dimensional spaces: i) similar examples with \( \Theta > 0 \) invertible but \( \Theta^{-1} \) unbounded!

i) possible \( \langle \phi_n, \psi_n \rangle \neq 0 \) for all \( n \) but \( \langle \phi_n, \psi_n \rangle \xrightarrow{n \to \infty} 0 \)!

i) unbounded \( \Theta \) or \( \Theta^{-1} \) always exist!

Moreover: i) physically relevant quantities are not preserved!

(continuous spectrum, pseudospectrum)

i) spectral instabilities!

concept of “unbounded metric” is trivial and physically doubtful
Mathematical frameworks to understand $\mathcal{P} \mathcal{T} H \mathcal{P} \mathcal{T} = H$ in a more general setting
Mathematical frameworks to understand $\mathcal{P}\mathcal{T} H \mathcal{P}\mathcal{T} = H$ in a more general setting than:

- $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ with $V(-x) = V(x)$
- $(\mathcal{P}\psi)(x) := \psi(-x)$, $(\mathcal{T}\psi)(x) := \overline{\psi(x)}$

**Remark.** In general, a $\mathcal{P}\mathcal{T}$-symmetric operator is not similar to a self-adjoint, normal or spectral operator.
Mathematical frameworks

to understand $\mathcal{P}\mathcal{T} H \mathcal{P}\mathcal{T} = H$ in a more general setting than:

- $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ with $V(-x) = V(x)$
- $(\mathcal{P}\psi)(x) := \psi(-x)$, $(\mathcal{T}\psi)(x) := \overline{\psi(x)}$

**Remark.** In general, a $\mathcal{P}\mathcal{T}$-symmetric operator is not similar to a self-adjoint, normal or spectral operator.

1. antilinear symmetry $[H, S] = 0$ with $S$ antiunitary (bijective and $\langle S\phi, S\psi \rangle = \langle \psi, \phi \rangle$)
e.g. $S := \mathcal{P}\mathcal{T}$
Mathematical frameworks

to understand $\mathcal{P}\mathcal{T} H \mathcal{P}\mathcal{T} = H$ in a more general setting than:

- $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ with $V(-x) = V(x)$
- $(\mathcal{P}\psi)(x) := \psi(-x)$, $(\mathcal{T}\psi)(x) := \overline{\psi(x)}$

**Remark.** In general, a $\mathcal{P}\mathcal{T}$-symmetric operator is not similar to a self-adjoint, normal or spectral operator.

1. **antilinear symmetry** $[H, S] = 0$ with $S$ antiunitary (bijective and $\langle S\phi, S\psi \rangle = \langle \psi, \phi \rangle$)

   e.g. $S := \mathcal{P}\mathcal{T}$

2. **self-adjointness in Krein spaces** $H$ is self-adjoint in an *indefinite* inner product space

   e.g. $\langle \cdot, \cdot \rangle := \langle P \cdot, \cdot \rangle$ after noticing $\mathcal{P}H\mathcal{P} = \mathcal{T}H\mathcal{T} = H^*$

   [Langer, Tretter 2004]
Mathematical frameworks to understand $\mathcal{P}\mathcal{T} H \mathcal{P}\mathcal{T} = H$ in a more general setting than:

- $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ with $V(-x) = V(x)$
- $(\mathcal{P}\psi)(x) := \psi(-x), \ (\mathcal{T}\psi)(x) := \overline{\psi(x)}$

**Remark.** In general, a $\mathcal{P}\mathcal{T}$-symmetric operator is not similar to a self-adjoint, normal or spectral operator.

1. **antilinear symmetry** $[H, S] = 0$ with $S$ antiunitary (bijective and $\langle S\phi, S\psi \rangle = \langle \psi, \phi \rangle$)

   e.g. $S := \mathcal{P}\mathcal{T}$

2. **self-adjointness in Krein spaces** $H$ is self-adjoint in an *indefinite* inner product space

   e.g. $[\cdot, \cdot] = \langle \cdot, \mathcal{P}\cdot \rangle$ after noticing $\mathcal{P}H\mathcal{P} = \mathcal{T}H\mathcal{T} = H^*$  

   [Langer, Tretter 2004]

3. **$J$-self-adjointness** $H^* = JHJ$ with $J$ conjugation (involutive and $\langle J\phi, J\psi \rangle = \langle \psi, \phi \rangle$)

   e.g. $J := \mathcal{T}$ after noticing $\mathcal{T}H\mathcal{T} = \mathcal{P}H\mathcal{P} = H^*$  

   [Borisov, D.K. 2007]
Mathematical frameworks to understand $\mathcal{P}\mathcal{T}H\mathcal{P}\mathcal{T} = H$ in a more general setting than:

- $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ with $\overline{V(-x)} = V(x)$
- $(\mathcal{P}\psi)(x) := \psi(-x)$, $(\mathcal{T}\psi)(x) := \overline{\psi(x)}$

**Remark.** In general, a $\mathcal{P}\mathcal{T}$-symmetric operator is *not* similar to a self-adjoint, normal or spectral operator.

1. **antilinear symmetry** $[H, S] = 0$ with $S$ *antiunitary* (bijective and $\langle S\phi, S\psi \rangle = \langle \psi, \phi \rangle$)
   
   e.g. $S := \mathcal{P}\mathcal{T}$

2. **self-adjointness in Krein spaces** $H$ is self-adjoint in an *indefinite* inner product space

   e.g. $[\cdot, \cdot] := \langle \cdot, \mathcal{P} \cdot \rangle$ after noticing $\mathcal{P}H\mathcal{P} = \mathcal{T}H\mathcal{T} = H^*$  
   
   [Langer, Tretter 2004]

3. **$J$-self-adjointness** $H^* = JHJ$ with $J$ *conjugation* (involutive and $\langle J\phi, J\psi \rangle = \langle \psi, \phi \rangle$)

   e.g. $J := \mathcal{T}$ after noticing $\mathcal{T}H\mathcal{T} = \mathcal{P}H\mathcal{P} = H^*$  
   
   [Borisov, D.K. 2007]

**Remark.** In general (in $\infty$-dimensional spaces), all the classes of operators are *unrelated*.

[Siegl 2008]
¿ Physical relevance?
¿ Physical relevance?

**Suggestions:**

- nuclear physics  [Scholtz, Geyer, Hahne 1992]
- optics  [Klaiman, Günther, Moiseyev 2008], [Schomerus 2010], [West, Kottos, Prosen 2010]
- solid state physics  [Bendix, Fleischmann, Kottos, Shapiro 2009]
- superconductivity  [Rubinstein, Sternberg, Ma 2007]
- electromagnetism  [Ruschhaupt, Delgado, Muga 2005], [Mostafazadeh 2009]

**Experiments:**

- optics  [Guo et al. 2009], [Longhi 2009], [Rüter et al. 2010]
- mechanics  [Bender, Berntson, Parker, Samuel 2012]
¿ Physical relevance? 

**suggestions:**
- nuclear physics [Scholtz, Geyer, Hahne 1992]
- optics [Klaiman, Günther, Moiseyev 2008], [Schomerus 2010], [West, Kottos, Prosen 2010]
- solid state physics [Bendix, Fleischmann, Kottos, Shapiro 2009]
- superconductivity [Rubinstein, Sternberg, Ma 2007]
- electromagnetism [Ruschhaupt, Delgado, Muga 2005], [Mostafazadeh 2009]

**experiments:**
- optics [Guo et al. 2009], [Longhi 2009], [Rüter et al. 2010]
- mechanics [Bender, Berntson, Parker, Samuel 2012]

¡ but !

“So far, there have been no experiments that prove clearly and definitively that quantum systems defined by non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians do exist in nature.” [Bender 2007]
The simplest $\mathcal{PT}$-symmetric model

$\mathcal{H} := L^2(-\frac{\pi}{2}, \frac{\pi}{2})$

$H_\alpha \psi := -\psi''$, $D(H_\alpha) := \left\{ \psi \in W^{2,2}(-\frac{\pi}{2}, \frac{\pi}{2}) \mid \psi'(\pm \frac{\pi}{2}) + i\alpha \psi(\pm \frac{\pi}{2}) = 0 \right\}$, $\alpha \in \mathbb{R}$

$-\Delta$

$\frac{d\psi}{dn} - i\alpha \psi = 0$ $\frac{d\psi}{dn} + i\alpha \psi = 0$
The simplest $\mathcal{PT}$-symmetric model

$\mathcal{H} := L^2\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$H_\alpha \psi := -\psi''$, $D(H_\alpha) := \left\{ \psi \in W^{2,2}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \mid \psi'(\pm \frac{\pi}{2}) + i\alpha \psi(\pm \frac{\pi}{2}) = 0 \right\}$, $\alpha \in \mathbb{R}$

$-\Delta$

$\frac{d\psi}{dn} - i\alpha \psi = 0$ $\frac{d\psi}{dn} + i\alpha \psi = 0$

**Theorem 1.** $H_\alpha$ is an $m$-sectorial operator with compact resolvent satisfying

$H_\alpha^* = H_{-\alpha} = \mathcal{T}H_\alpha \mathcal{T}$  \hspace{1cm} ($\mathcal{T}$-self-adjointness)
The simplest $\mathcal{PT}$-symmetric model

$\mathcal{H} := L^2(-\frac{\pi}{2}, \frac{\pi}{2})$

$H_\alpha \psi := -\psi''$, $D(H_\alpha) := \left\{ \psi \in W^{2,2}(-\frac{\pi}{2}, \frac{\pi}{2}) \mid \psi'(\pm \frac{\pi}{2}) + i\alpha \psi(\pm \frac{\pi}{2}) = 0 \right\}$, $\alpha \in \mathbb{R}$

$D := \{-\Delta\}$

$\frac{d\psi}{dn} - i\alpha \psi = 0$ \hspace{1cm} $\frac{d\psi}{dn} + i\alpha \psi = 0$

**Theorem 1.** $H_\alpha$ is an $m$-sectorial operator with compact resolvent satisfying

$H_\alpha^* = H_{-\alpha} = \mathcal{T} H_\alpha \mathcal{T}$

($\mathcal{T}$-self-adjointness)

**Theorem 2.** $\sigma(H_\alpha) = \{\alpha^2\} \cup \{n^2\}_{n=1}^\infty$
The simplest $\mathcal{PT}$-symmetric model

$\mathcal{H} := L^2(-\frac{\pi}{2}, \frac{\pi}{2})$

$H_\alpha \psi := -\psi''$, $D(H_\alpha) := \left\{ \psi \in W^{2,2}(-\frac{\pi}{2}, \frac{\pi}{2}) \mid \psi'(\pm \frac{\pi}{2}) + i\alpha \psi(\pm \frac{\pi}{2}) = 0 \right\}$, $\alpha \in \mathbb{R}$

$\begin{align*}
-H^2 & \\
\frac{d\psi}{dn} - i\alpha \psi &= 0 & \frac{d\psi}{dn} + i\alpha \psi &= 0
\end{align*}$

**Theorem 1.** $H_\alpha$ is an $m$-sectorial operator with compact resolvent satisfying

$H^* = H_{-\alpha} = \mathcal{T} H_\alpha \mathcal{T}$  

($\mathcal{T}$-self-adjointness)

**Theorem 2.** $\sigma(H_\alpha) = \{\alpha^2\} \cup \{n^2\}_{n=1}^\infty$

**Corollary.** The spectrum of $H_\alpha$ is

$\begin{align*}
\text{always real,} & \\
\text{simple if } \alpha \notin \mathbb{Z} \setminus \{0\}. & \\
\end{align*}$
Scattering realisation in QM

[Hernandez-Coronado, D.K., Siegl 2011]

**scattering** by a compactly supported *even* potential $V$:

$$-\psi'' + V\psi = k^2\psi \quad k > 0$$

$$\psi_{in}(x) = e^{ikx} + Re^{-ikx} \quad \quad \psi_{out}(x) = T e^{ikx}$$
**Scattering realisation in QM**

[Hernandez-Coronado, D.K., Siegl 2011]

**scattering** by a compactly supported *even* potential $V$:

$-\psi'' + V\psi = k^2\psi \quad k > 0$

\[\psi_{\text{in}}(x) = e^{ikx} + \text{Re}^{-ikx} \quad \psi_{\text{out}}(x) = T e^{ikx}\]

**perfect transmission** (i.e. $R = 0$)

\[\begin{cases} 
-\psi'' + V\psi = k^2\psi & \text{in } (0, \pi) \\
\psi' - ik\psi = 0 & \text{at } 0, \pi
\end{cases}\]

\[\therefore \quad \text{non-linear}\]
scattering by a compactly supported even potential $V$: $-\psi'' + V\psi = k^2\psi$ $k > 0$

$\psi_{in}(x) = e^{ikx} + Re^{-ikx}$

$\psi'' + V\psi = \mu(\alpha)\psi$ in $(0, \pi)$

$\psi' - ik\psi = 0$ at $0, \pi$

solutions given by a non-self-adjoint $\mathcal{PT}$-symmetric spectral problem:

$-\psi'' + V\psi = \mu(\alpha)\psi$ in $(0, \pi)$

$\psi' + i\alpha\psi = 0$ at $0, \pi$

$\mu(\alpha) = \alpha^2$

perfect transmission $(i.e. R = 0)$

non-linear

don't-hermitian
The metric operator

[D.K., Siegl, Železný 2011]

**Theorem 3.** Let $\alpha \not\in \mathbb{Z} \setminus \{0\}$.

Then $H_\alpha$ is similar to a self-adjoint operator $h_\alpha := \Omega H_\alpha \Omega^{-1}$ with the metric

$$\Theta := \Omega^* \Omega = I + K$$

$$K(x, y) = \frac{2i}{\pi} e^{i \frac{\alpha}{2} (x-y)} \sin \left( \frac{\alpha}{2} (x - y) \right) + \frac{i\alpha}{\pi} \left( |y - x| - \pi \right) \text{sgn}(y - x) + \frac{\alpha^2}{\pi} \left( \frac{\pi^2}{4} - xy - \frac{\pi}{2} |y - x| \right)$$
Theorem 3. Let $\alpha \not\in \mathbb{Z} \setminus \{0\}$.

Then $H_\alpha$ is similar to a self-adjoint operator $h_\alpha := \Omega H_\alpha \Omega^{-1}$ with the metric

\[ \Theta := \Omega^* \Omega = I + K \]

\[ K(x, y) = \frac{2i}{\pi} e^{i \frac{\alpha}{2} (x - y)} \sin \left( \frac{\alpha}{2} (x - y) \right) + \frac{i\alpha}{\pi} \left( |y - x| - \pi \right) \text{sgn}(y - x) + \frac{\alpha^2}{\pi} \left( \frac{\pi^2}{4} - xy - \frac{\pi}{2} |y - x| \right) \]

Moreover,

\[ h_\alpha = -\Delta_N + \alpha^2 \chi_0^N \langle \chi_0^N, \cdot \rangle \]

\[ \chi_0^N(x) := \pi^{-1/2} \] (Neumann ground state)

non-Hermitian $H_\alpha \leftrightarrow$ non-local $h_\alpha$
Theorem 3. Let $\alpha \notin \mathbb{Z} \setminus \{0\}$.

Then $H_\alpha$ is similar to a self-adjoint operator $h_\alpha := \Omega H_\alpha \Omega^{-1}$ with the metric

$$\Theta := \Omega^* \Omega = I + K$$

$$K(x, y) = \frac{2i}{\pi} e^{i \frac{\alpha}{2} (x-y)} \sin \left( \frac{\alpha}{2} (x-y) \right) + \frac{i \alpha}{\pi} \left( |y-x| - \pi \right) \text{sgn}(y-x) + \frac{\alpha^2}{\pi} \left( \frac{\pi^2}{4} - xy - \frac{\pi}{2} |y-x| \right)$$

Moreover,

$$h_\alpha = -\Delta_N + \alpha^2 \chi_0^N \langle \chi_0^N , \cdot \rangle$$

$\chi_0^N(x) := \pi^{-1/2}$ (Neumann ground state)

non-Hermitian $H_\alpha \leftrightarrow$ non-local $h_\alpha$


$$E_n = \begin{cases} \alpha^2 \quad &\text{if } n = 0 \\ n^2 = E_n^N = E_n^D \quad &\text{if } n \geq 1 \end{cases}$$

$$\phi_n(x) = \begin{cases} \frac{1}{\sqrt{\pi}} e^{i \alpha (x+\frac{\pi}{2})} \quad &\text{if } n = 0 \\ \chi_n^N(x) + i \frac{\alpha}{n} \chi_n^D(x) \quad &\text{if } n \geq 1 \end{cases}$$

$$\Omega := \sum_{n=0}^\infty \chi_n^N \langle \phi_n , \cdot \rangle = \chi_0^N \langle \phi_0 , \cdot \rangle - \chi_0^N \langle \chi_0^N , \cdot \rangle + \sum_{n=0}^\infty \chi_n^N \langle \chi_n^N , \cdot \rangle - i \alpha \sum_{n=1}^\infty \frac{1}{n} \chi_n^N \langle \chi_n^D , \cdot \rangle$$

$$= \chi_0^N \langle \phi_0 , \cdot \rangle - \chi_0^N \langle \chi_0^N , \cdot \rangle + I + \alpha p (-\Delta_D)^{-1}$$

($-\Delta_D = p^* p$)
Definition ([Bender, Brody Jones 2002], [Albeverio, Kuzhel 2005]).
Let $H$ be $\mathcal{P}$-self-adjoint and $\mathcal{C}$ bounded.

$H$ is $\mathcal{C}$-symmetric $\iff$

\[
\begin{align*}
[H, \mathcal{C}] &= 0 \\
\mathcal{C}^2 &= I \\
\mathcal{P}\mathcal{C} &\text{ is a metric for } H
\end{align*}
\]
The $\mathcal{C}$ operator

**Definition** ([Bender, Brody Jones 2002], [Albeverio, Kuzhel 2005]).

Let $H$ be $\mathcal{P}$-self-adjoint and $\mathcal{C}$ bounded.

$$H \text{ is } \mathcal{C}\text{-symmetric} \iff \begin{cases} [H, \mathcal{C}] = 0 \\ \mathcal{C}^2 = I \\ \mathcal{P}\mathcal{C} \text{ is a metric for } H \end{cases}$$

Thus: $\mathcal{C} := \mathcal{P}\Theta$ for $\Theta$ satisfying $(\mathcal{P}\Theta)^2 = I$
The $\mathcal{C}$ operator

**Definition** ([Bender, Brody Jones 2002], [Albeverio, Kuzhel 2005]).

Let $H$ be $\mathcal{P}$-self-adjoint and $\mathcal{C}$ bounded.

$H$ is $\mathcal{C}$-symmetric \iff \[
\begin{cases}
[H, \mathcal{C}] = 0 \\
\mathcal{C}^2 = I \\
\mathcal{P}\mathcal{C} \text{ is a metric for } H
\end{cases}
\]

Thus: $\mathcal{C} := \mathcal{P}\Theta$ for $\Theta$ satisfying $(\mathcal{P}\Theta)^2 = I$

Our explicit result:

$\mathcal{C} = \mathcal{P} + L$ \quad with

$L(x, y) = \alpha e^{-i\alpha(y+x)} \left[ \tan(\alpha\frac{\pi}{2}) - i \text{sgn}(y + x) \right]$ \quad ($|\alpha| < 1$)
General $\mathcal{PT}$-symmetric case

[D.K., Siegl 2010]

$H_{\alpha,\beta} \psi := -\psi''$, $D(H_{\alpha,\beta}) := \left\{ \psi \in W^{2,2}(-\frac{\pi}{2}, \frac{\pi}{2}) \mid \psi'\left(\pm\frac{\pi}{2}\right) + (i\alpha \pm \beta)\psi\left(\pm\frac{\pi}{2}\right) = 0 \right\}$

$\beta > 0$
$\beta = 0$
$\beta < 0$
\( H_{\alpha,\beta} \psi := -\psi'' \), \( D(H_{\alpha,\beta}) := \left\{ \psi \in W^{2,2}(-\frac{\pi}{2}, \frac{\pi}{2}) \mid \psi'(\pm \frac{\pi}{2}) + (i\alpha \pm \beta) \psi(\pm \frac{\pi}{2}) = 0 \right\} \)

\[ \Theta = I + K \quad \text{with} \]
\[ K(x, y) = e^{i\alpha(x-y)-\beta|x-y|} \left[ c + i\alpha \text{sgn}(x-y) \right] \quad c \in \mathbb{R} \]

\( \Theta > 0 \ \text{if} \ \beta > 0 \ \text{large or} \ c^2 + \alpha^2 \ \text{small} \)
Imaginary cubic oscillator

\[ H = -\frac{d^2}{dx^2} + ix^3 \]
on \[ L^2(\mathbb{R}) \]
Imaginary cubic oscillator

[D.K., Siegl 2012]

\[ H = -\frac{d^2}{dx^2} + ix^3 \]  
on \( L^2(\mathbb{R}) \),  
\( D(H) := \{ \psi \in L^2(\mathbb{R}) \mid H\psi \in L^2(\mathbb{R}) \} \)

- \( H \) is m-accretive \( \Rightarrow \sigma(H) \geq 0 \)  
  [Edmunds, Evans 1987]

- \( H \) has purely discrete spectrum  
  [Caliceti, Graffi, Maioli 1980], [Mezinescu 2001]

- all eigenvalues of \( H \) are real  
  [Dorey, Dunning, Tateo 2001], [Shin 2002]
Imaginary cubic oscillator  
[D.K., Siegl 2012]

\[ H = -\frac{d^2}{dx^2} + ix^3 \]
on \( L^2(\mathbb{R}) \),  
\[ D(H) := \{ \psi \in L^2(\mathbb{R}) \mid H\psi \in L^2(\mathbb{R}) \} \]

- \( H \) is m-accretive  \( \Rightarrow \)  \( \Re \sigma(H) \geq 0 \)  
  [Edmunds, Evans 1987]
- \( H \) has purely discrete spectrum  
  [Caliceti, Graffi, Maioli 1980], [Mezinescu 2001]
- all eigenvalues of \( H \) are real  
  [Dorey, Dunning, Tateo 2001], [Shin 2002]

¿ Does \( H \) possess metric ?
Imaginary cubic oscillator

[D.K., Siegl 2012]

\[ H = -\frac{d^2}{dx^2} + ix^3 \]

on \( L^2(\mathbb{R}) \), \( D(H) := \{ \psi \in L^2(\mathbb{R}) \mid H\psi \in L^2(\mathbb{R}) \} \)

- \( H \) is m-accretive \( \Rightarrow \Re \sigma(H) \geq 0 \) [Edmunds, Evans 1987]
- \( H \) has purely discrete spectrum [Caliceti, Graffi, Maioli 1980], [Mezinescu 2001]
- all eigenvalues of \( H \) are real [Dorey, Dunning, Tateo 2001], [Shin 2002]
- Does \( H \) possess metric?

- eigenfunctions of \( H \) form a complete set
Imaginary cubic oscillator

[D.K., Siegl 2012]

\[ H = -\frac{d^2}{dx^2} + ix^3 \]

on \( L^2(\mathbb{R}) \), \( D(H) := \{ \psi \in L^2(\mathbb{R}) \mid H\psi \in L^2(\mathbb{R}) \} \)

- \( H \) is m-accretive \( \Rightarrow \Re \sigma(H) \geq 0 \) [Edmunds, Evans 1987]
- \( H \) has purely discrete spectrum [Caliceti, Graffi, Maioli 1980], [Mezinescu 2001]
- all eigenvalues of \( H \) are real [Dorey, Dunning, Tateo 2001], [Shin 2002]

¿ Does \( H \) possess metric ?

- eigenfunctions of \( H \) form a complete set

Theorem. Metric operator for \( H \) does not exist.
Imaginary cubic oscillator

[D.K., Siegl 2012]

\[
H = -\frac{d^2}{dx^2} + ix^3
\]
on \quad \mathcal{L}^2(\mathbb{R}), \quad D(H) := \{\psi \in \mathcal{L}^2(\mathbb{R}) \mid H\psi \in \mathcal{L}^2(\mathbb{R})\}

- \quad H \text{ is m-accretive } \Rightarrow \Re \sigma(H) \geq 0 \quad [\text{Edmunds, Evans 1987}]
- \quad H \text{ has purely discrete spectrum} \quad [\text{Caliceti, Graffi, Maioli 1980}, \text{Mezinescu 2001}]
- \quad \text{all eigenvalues of } H \text{ are real} \quad [\text{Dorey, Dunning, Tateo 2001}, \text{Shin 2002}]

¿ Does $H$ possess metric? 

- \quad \text{eigenfunctions of } H \text{ form a complete set}

Theorem. \quad \text{Metric operator for } H \text{ does not exist.}

Proof.

- \quad \text{Let metric exist } \Rightarrow \quad \| (H - z)^{-1} \| \leq \frac{C}{|\Re z|}, \quad \forall z \in \mathbb{C}, \; \Im z \neq 0
Imaginary cubic oscillator

[D.K., Siegl 2012]

\[ H = -\frac{d^2}{dx^2} + ix^3 \quad \text{on} \quad L^2(\mathbb{R}), \quad D(H) := \{ \psi \in L^2(\mathbb{R}) \mid H\psi \in L^2(\mathbb{R}) \} \]

- \( H \) is m-accretive \( \Rightarrow \) \( \Re \sigma(H) \geq 0 \) \hfill [Edmunds, Evans 1987]
- \( H \) has purely discrete spectrum \hfill [Caliceti, Graffi, Maioli 1980], [Mezinescu 2001]
- all eigenvalues of \( H \) are real \hfill [Dorey, Dunning, Tateo 2001], [Shin 2002]

¿ Does \( H \) possess metric? \hfill \[ \text{Metric operator for } H \text{ does not exist.} \]

Theorem. \hfill \[ \text{Metric operator for } H \text{ does not exist.} \]

Proof.

- Let metric exist \( \Rightarrow \) \( \| (H - z)^{-1} \| \leq \frac{C}{|\Im z|}, \quad \forall z \in \mathbb{C}, \quad \Im z \neq 0 \)
- \( \| (H - \sigma z)^{-1} \| = \sigma^{-1} \| (H_h - z)^{-1} \| \) \quad with \quad \( H_h := -h^2 \frac{d^2}{dx^2} + ix^3, \quad h := \sigma^{-5/6} \)
Imaginary cubic oscillator

[D.K., Siegl 2012]

\[ H = -\frac{d^2}{dx^2} + ix^3 \]
on \[ L^2(\mathbb{R}), \quad D(H) := \{ \psi \in L^2(\mathbb{R}) \mid H\psi \in L^2(\mathbb{R}) \} \]

- \( H \) is m-accretive \( \Rightarrow \Re \sigma(H) \geq 0 \) \[ \text{[Edmunds, Evans 1987]} \]
- \( H \) has purely discrete spectrum \[ \text{[Caliceti, Graffi, Maioli 1980], [Mezinescu 2001]} \]
- all eigenvalues of \( H \) are real \[ \text{[Dorey, Dunning, Tateo 2001], [Shin 2002]} \]

¿ Does \( H \) possess metric ?

- eigenfunctions of \( H \) form a complete set

**Theorem.** Metric operator for \( H \) does not exist.

**Proof.**
- Let metric exist \( \Rightarrow \| (H - z)^{-1} \| \leq \frac{C}{|\Im z|}, \quad \forall z \in \mathbb{C}, \Im z \neq 0 \)

- \( \| (H - \sigma z)^{-1} \| = \sigma^{-1} \| (H_h - z)^{-1} \| \quad \text{with} \quad H_h := -h^2 \frac{d^2}{dx^2} + ix^3, \quad h := \sigma^{-5/6} \)

- \( \| (H_h - z)^{-1} \| = O(h^{-n}), \quad \forall n > 0 \quad \Rightarrow \quad \text{contradiction} \quad q.e.d. \)  
  \[ \text{[Davies 1999]} \]
Pseudospectra and $\mathcal{PT}$-symmetry

[work in progress with Siegl and Tater]
Pseudospectra and $\mathcal{PT}$-symmetry

[work in progress with Siegl and Tater]

$\sigma_\varepsilon(H) := \{ z \in \mathbb{C} \mid \|(H - z)^{-1}\| > \varepsilon^{-1} \}$

[Trefethen, Embree 2005], [Davies 2007]
Pseudospectra and $\mathcal{PT}$-symmetry

[work in progress with Siegl and Tater]

$$\sigma_\varepsilon(H) := \{z \in \mathbb{C} \mid \|(H - z)^{-1}\| > \varepsilon^{-1}\}$$  

[Trefethen, Embree 2005], [Davies 2007]

- $H$ is self-adjoint $\implies \|(H - z)^{-1}\| = \frac{1}{\text{dist}(z, \sigma(H))} \implies$ trivial pseudospectrum
Pseudospectra and $\mathcal{PT}$-symmetry

[work in progress with Siegl and Tater]

$$\sigma_\varepsilon(H) := \{ z \in \mathbb{C} \mid \|(H - z)^{-1}\| > \varepsilon^{-1} \}$$

[Trefethen, Embree 2005], [Davies 2007]

- $H$ is self-adjoint \implies \|(H - z)^{-1}\| = \frac{1}{\text{dist}(z, \sigma(H))} \implies$ trivial pseudospectrum

- For non-self-adjoint operators, pseudospectra more relevant than spectra!
Pseudospectra and $\mathcal{PT}$-symmetry

[work in progress with Siegl and Tater]

\( \sigma_\varepsilon(H) := \{ z \in \mathbb{C} \mid \|(H - z)^{-1}\| > \varepsilon^{-1} \} \)  

[Trefethen, Embree 2005], [Davies 2007]

- \( H \) is self-adjoint \( \implies \|(H - z)^{-1}\| = \frac{1}{\text{dist}(z, \sigma(H))} \implies \) trivial pseudospectrum

- For non-self-adjoint operators, pseudospectra more relevant than spectra!

- \( \sigma_\varepsilon(H) = \bigcup_{\|V\| < \varepsilon} \sigma(H + V) \implies \) spectral instabilities
Pseudospectra and $\mathcal{PT}$-symmetry

[work in progress with Siegl and Tater]

$$\sigma_\varepsilon(H) := \{ z \in \mathbb{C} \mid \| (H - z)^{-1} \| > \varepsilon^{-1} \}$$

[Trefethen, Embree 2005], [Davies 2007]

- $H$ is self-adjoint $\implies \| (H - z)^{-1} \| = \frac{1}{\text{dist}(z, \sigma(H))} \implies$ trivial pseudospectrum

- For non-self-adjoint operators, pseudospectra more relevant than spectra!

- $\sigma_\varepsilon(H) = \bigcup \sigma(H + V) \implies$ spectral instabilities

- $\exists$ metric $\implies$ trivial pseudospectrum
Conclusions

Ad \( PT \)-symmetry:

→ no extension of QM
→ rather an alternative (quasi-Hermitian) representation
→ overlooked for over 70 years

some rigorous treatments still missing!
Conclusions

Ad $\mathcal{P}\mathcal{T}$-symmetry:

$\rightarrow$ no extension of QM
$\rightarrow$ rather an alternative (quasi-Hermitian) representation
$\rightarrow$ overlooked for over 70 years

$\downarrow$ some rigorous treatments still missing!

Ad our model:

$\rightarrow$ shamefully simple
$\rightarrow$ closed formulae for the spectrum, metric operator, self-adjoint counterpart, etc.
$\rightarrow$ rigorous treatment

$\downarrow$ physical relevance!
Conclusions

Ad $\mathcal{PT}$-symmetry:

→ no extension of QM
→ rather an alternative (quasi-Hermitian) representation
→ overlooked for over 70 years

¡ some rigorous treatments still missing !

Ad our model:

→ shamefully simple
→ closed formulae for the spectrum, metric operator, self-adjoint counterpart, etc.
→ rigorous treatment

¡ physical relevance !

Ad $ix^3$:

¡ metric operator does not exist !
(bad basicity properties, non-trivial pseudospectrum, spectral instabilities)
Collection of open problems

ESF exploratory workshop on

Mathematical aspects of the physics with non-self-adjoint operators

30 August - 3 September 2010
Prague, Czech Republic

http://www.ujf.cas.cz/ESFxNSA/

Studying non-self-adjoint operators is like being a vet rather than a doctor: one has to acquire a much wider range of knowledge, and to accept that one cannot expect to have as high a rate of success when confronted with particular cases.

E. B. Davies 2007

Convenors: J.-P. Gazeau (Paris), D. Krejcirik (Prague), P. Siegl (Prague)

Some of the open problems also available in Integral Equations Operator Theory.


