Non-Hermitian operators in quantum physics

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First, there's the room you can see through the glass – that's just the same as our drawing-room, only the things go the other way.

Hors de ligne

(Outline)

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1. QM with non-Hermitian operators

   (just some conceptual remarks)

2. $\mathcal{PT}$-symmetry

   (what is known and my point of view)

3. physical $\mathcal{PT}$-symmetric models in QM

   (non-self-adjoint Robin boundary conditions)

4. imaginary cubic oscillator

   (about the non-existence of the metric operator)

5. Conclusions

¿ QM with non-Hermitian operators ?

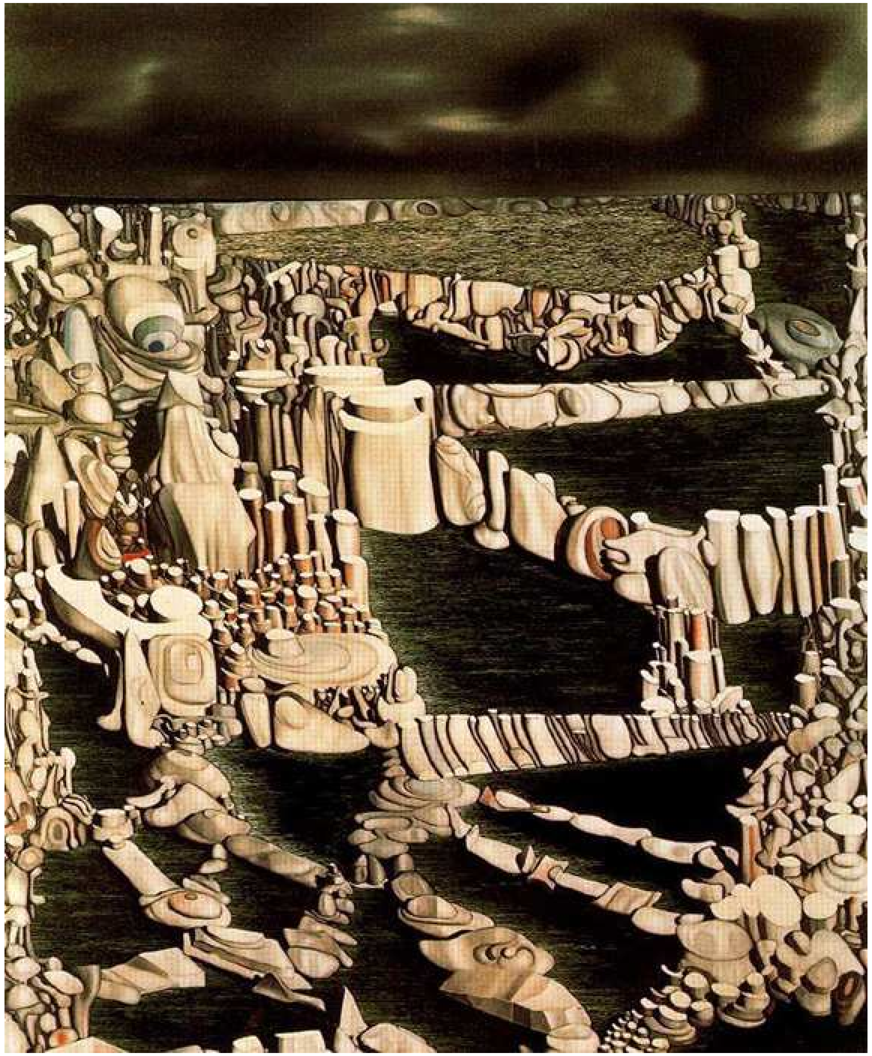
$$\mathbb{C}$$

$$\mathbb{R}$$

$$\mathbb{I}$$

$$H^* = H$$

$$H^{\mathcal{PT}} = H$$



Imaginary Numbers by Yves Tanguy, 1954

(Museo Thyssen-Bornemisza, Madrid)

Insignificant non-Hermiticity

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Theorem (spectral theorem).
Let $H = H^*$. Then

$$\boxed{f(H) = \int_{\sigma(H)} f(\lambda)\; \mathrm{d}E_H(\lambda)}$$

for any complex-valued continuous function f.

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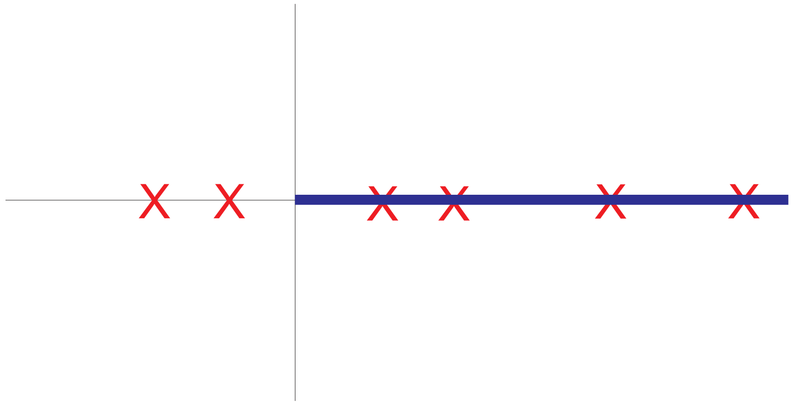
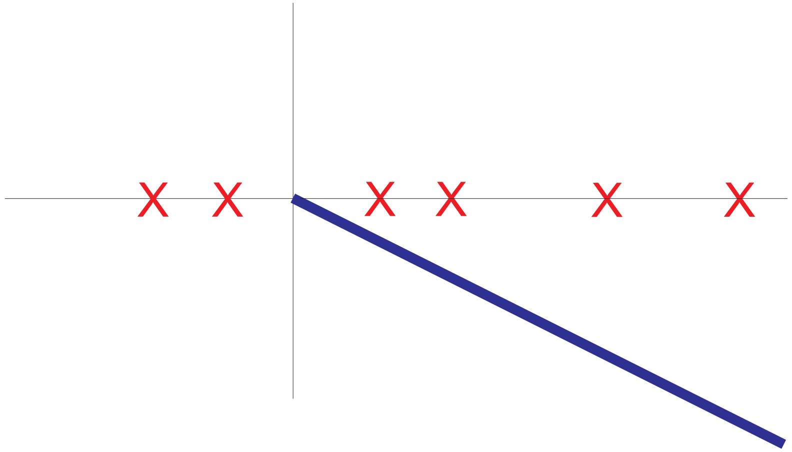
$\theta = 0$



$\Im\theta > 0$



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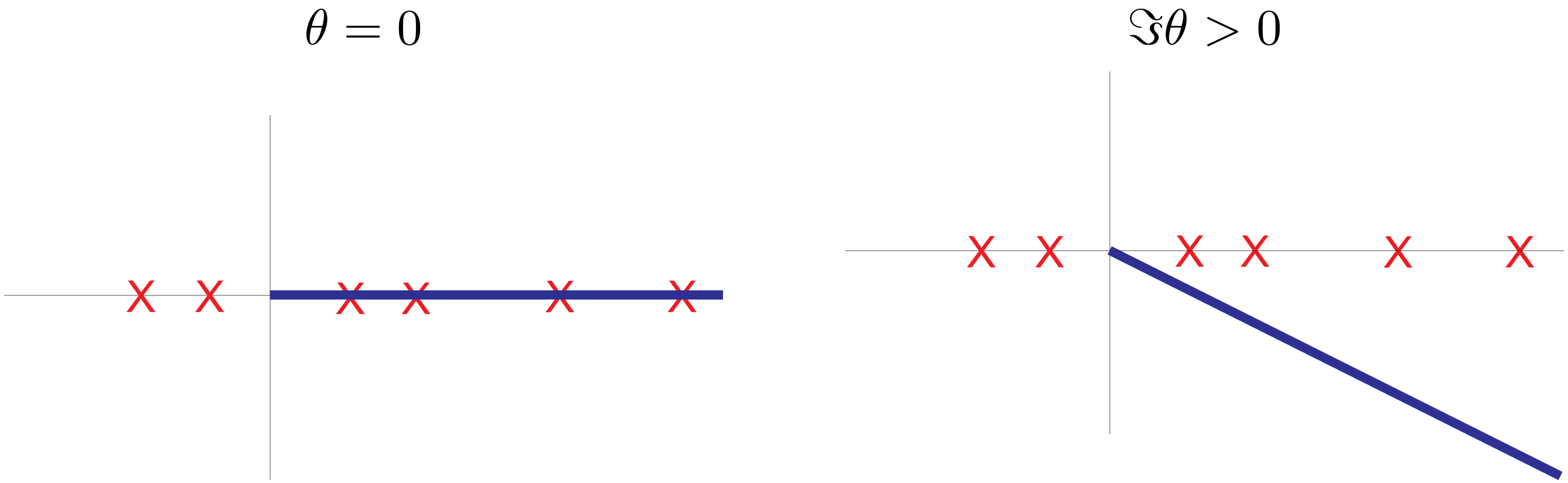

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Example 2. adiabatic transition probability for $H(t) := \vec{\gamma}(t/\tau) \cdot \vec{\sigma}$, $\tau \to \infty$

[Berry 1990], [Joye, Kunz, Pfister 1991], [Jakšić, Segert 1993], ...

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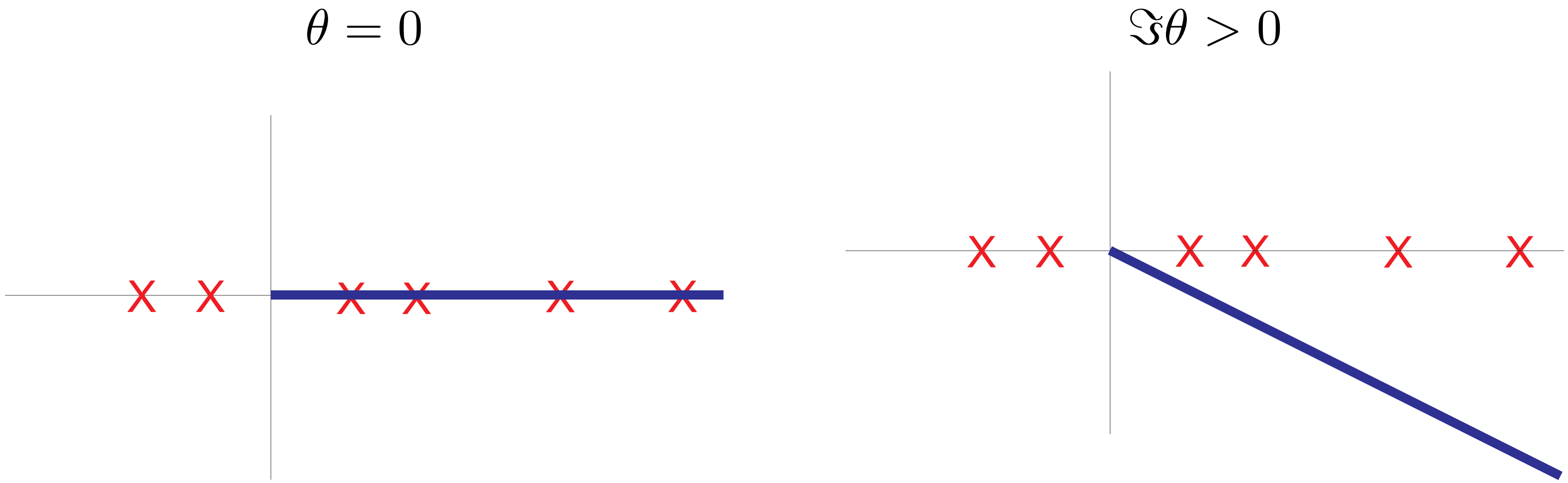

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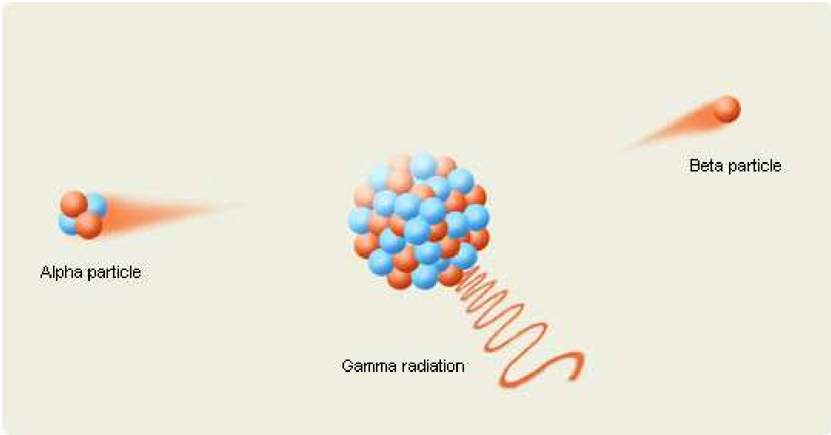
Example 3. Regge theory $H_l := -\dfrac{\mathrm{d}^2}{\mathrm{d}r^2} + \dfrac{l(l+1)}{r^2} + V(r), \quad l \in \mathbb{C}$

[Regge 1957], [Connor 1990], [Sokolovski 2011], ...

Approximate non-Hermiticity

open systems

Example 1. radioactive decay



Example 2. dissipative Schrödinger operators in semiconductor physics

Baro, Behrndt, Kaiser, Neidhardt, Rehberg, . . .

Example 3. repeated interaction quantum systems

Bruneau, Joye, Merkli, Pillet, . . .

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Unitary groups on *a Hilbert space* are generated by self-adjoint operators.

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Theorem (Stone's theorem).
Unitary groups on *a Hilbert space* are generated by self-adjoint operators.

¿ yes ?

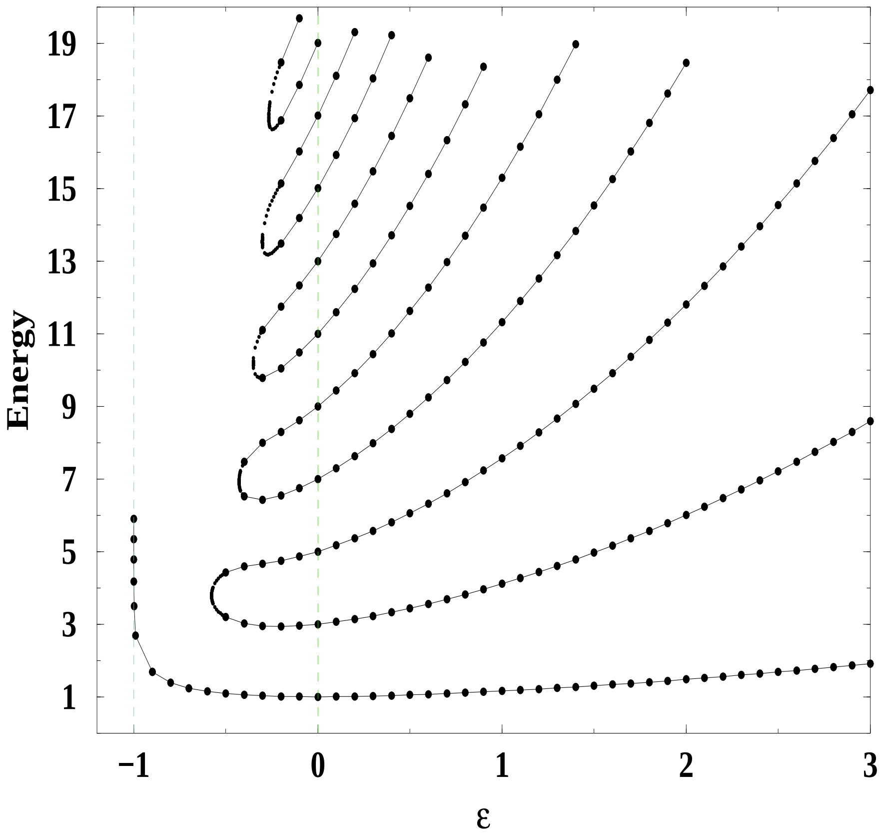
by changing the Hilbert space,
preserving a similarity to self-adjoint operators

Non-Hermitian Hamiltonians with real spectra

$$-\Delta + V \quad \text{in} \quad L^2(\mathbb{R})$$

$V(x) = x^2 + ix^3$ [Caliceti, Graffi, Maioli 1980]

$V(x) = x^2\,(ix)^\varepsilon$



[Bessis, Zinn-Justin]
[Bender, Boettcher 1998]
[Dorey, Dunning, Tateo 2001]
[Shin 2002]
[Azizov, Kuzhel, Günther, Trunk 2010]

$$V(x) = \begin{cases} i\,\mathrm{sgn}(x) & \text{if} \quad x \in (-L, L) \\ \infty & \text{elsewhere} \end{cases}$$

[Znojil 2001]

¿ What is behind the reality of the spectrum ?

$\mathcal{PT}$ - symmetry

$$[H, \mathcal{PT}] = 0$$

$$(\mathcal{P}\psi)(x) := \psi(-x)$$

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We have in mind $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ *with* $\overline{V(-x)} = V(x)$.

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Here we assume that H *has purely discrete spectrum.*

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perturbation-theory insight

$H_0 + \varepsilon V =: H \rightsquigarrow$ spectrum moves at most by $\varepsilon \|V\|$

self-adjoint $\nearrow$ $\nwarrow$ $\mathcal{PT}$-symmetric $\overset{*}{\Longrightarrow}$ simple eigenvalues remain real for small ε

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Moreover, let the eigenstates of H form a Riesz basis. $H\psi_n = E_n\psi_n$, $H^*\phi_n = E_n\phi_n$

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$$\underbrace{H_0}_{\text{self-adjoint}} + \varepsilon \underbrace{V}_{\mathcal{PT}\text{-symmetric}} =: H \rightsquigarrow \text{ spectrum moves at most by } \varepsilon \, \|V\|$$

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$\Longrightarrow$ $H^* = \Theta H \Theta^{-1}$ where $\boxed{\Theta := \sum_n \phi_n \langle \phi_n, \cdot \rangle}$ is positive, bounded, boundedly invertible

$\Longrightarrow$ H is self-adjoint in $\big(L^2, \langle \cdot, \underbrace{\Theta}_{\text{metric}} \cdot \rangle\big)$, *i.e.* $\Theta^{1/2} H \Theta^{-1/2}$ is self-adjoint in $\big(L^2, \langle \cdot, \cdot \rangle\big)$

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Albeverio-Fei-Kurasov, Bender-Brody-Jones, Caliceti-Graffi-Sjöstrand, Fring, Graefe-Schubert, Kretschmer-Szymanowski, Langer-Tretter, Mostafazadeh, Scholtz-Geyer-Hahne, Znojil, . . .

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concept of "unbounded metric" is trivial and physically doubtful

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Remark. In general (in ∞-dimensional spaces), all the classes of operators are unrelated.

[Siegl 2008]

¿ Physical relevance ?

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suggestions:

- nuclear physics [Scholtz, Geyer, Hahne 1992]
- optics [Klaiman, Günther, Moiseyev 2008], [Schomerus 2010], [West, Kottos, Prosen 2010]
- solid state physics [Bendix, Fleischmann, Kottos, Shapiro 2009]
- superconductivity [Rubinstein, Sternberg, Ma 2007]
- electromagnetism [Ruschhaupt, Delgado, Muga 2005], [Mostafazadeh 2009]

experiments:

- optics [Guo *et al.* 2009], [Longhi 2009], [Rüter *et al.* 2010]
- mechanics [Bender, Berntson, Parker, Samuel 2012]

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¡ but !

"So far, there have been no experiments that prove clearly and definitively that quantum systems defined by non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians do exist in nature."

[Bender 2007]

The simplest $\mathcal{PT}$ - symmetric model

[D.K., Bíla, Znojil 2006]

$\mathcal{H} := L^2(-\frac{\pi}{2}, \frac{\pi}{2})$

$$H_\alpha \psi := -\psi'' , \quad D(H_\alpha) := \left\{ \psi \in W^{2,2}(-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \;\middle|\; \psi'(\pm\tfrac{\pi}{2}) + i\alpha\,\psi(\pm\tfrac{\pi}{2}) = 0 \right\}, \quad \alpha \in \mathbb{R}$$

$\frac{d\psi}{dn} - i\alpha\psi = 0$ $\qquad$ $\frac{d\psi}{dn} + i\alpha\psi = 0$

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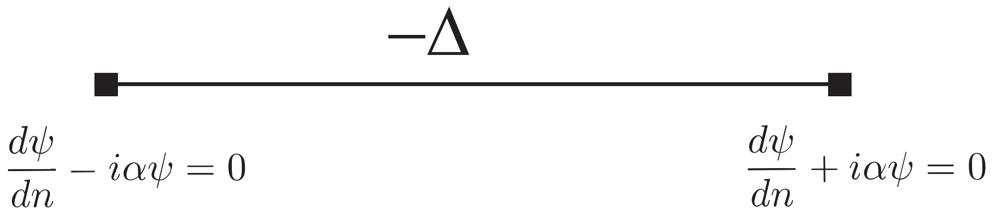

Theorem 1. H_α is an m-sectorial operator with compact resolvent satisfying

$$\boxed{H_\alpha^* = H_{-\alpha} = \mathcal{T} H_\alpha \mathcal{T}} \qquad (\mathcal{T}\text{-self-adjointness})$$

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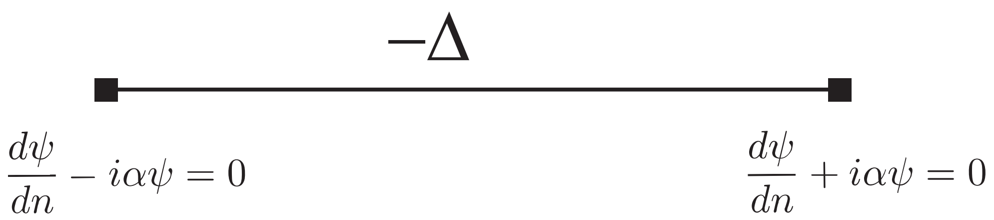
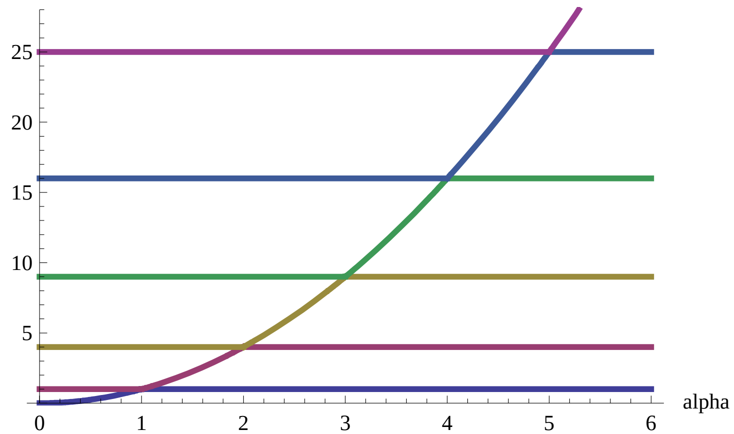


Theorem 1. H_α is an m-sectorial operator with compact resolvent satisfying

$$\boxed{H_\alpha^* = H_{-\alpha} = \mathcal{T} H_\alpha \mathcal{T}} \qquad (\mathcal{T}\text{-self-adjointness})$$

Theorem 2. $\boxed{\sigma(H_\alpha) = \{\alpha^2\} \cup \{n^2\}_{n=1}^{\infty}}$



The simplest $\mathcal{PT}$- symmetric model

[D.K., Bíla, Znojil 2006]

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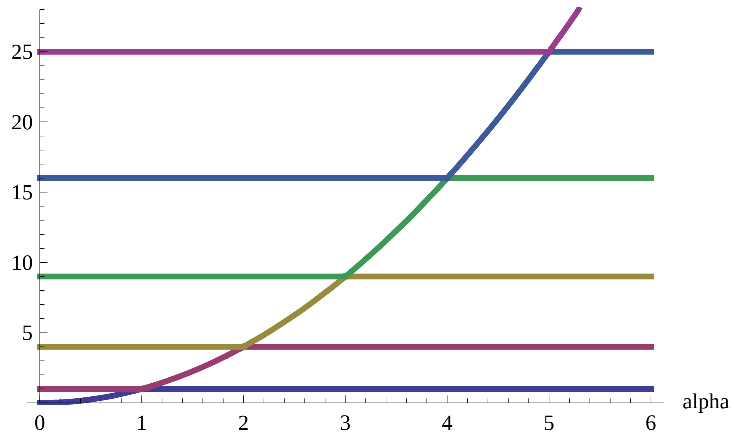
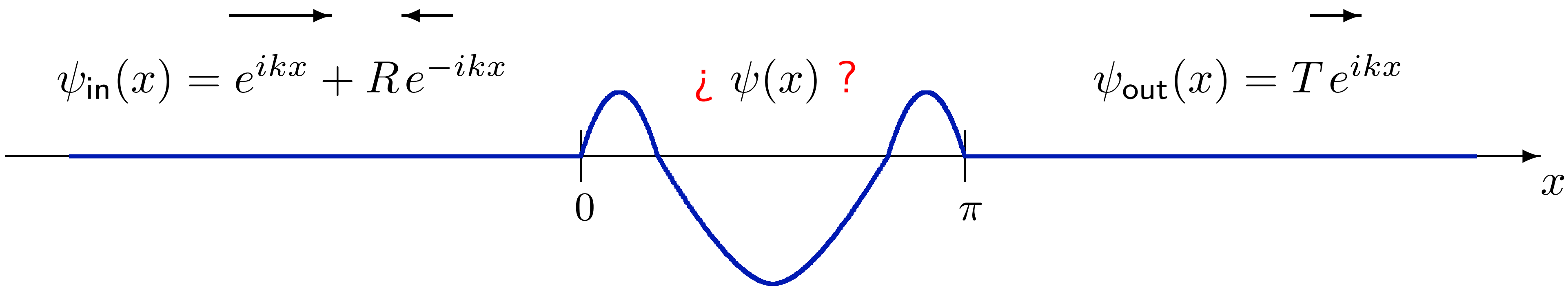


Corollary. The spectrum of H_α is $\begin{cases} \text{always real,} \\ \text{simple if } \ \alpha \notin \mathbb{Z} \setminus \{0\}. \end{cases}$

Scattering realisation in QM

[Hernandez-Coronado, D.K., Siegl 2011]

scattering by a compactly supported *even* potential V: $-\psi'' + V\psi = k^2\psi$ $\quad k > 0$



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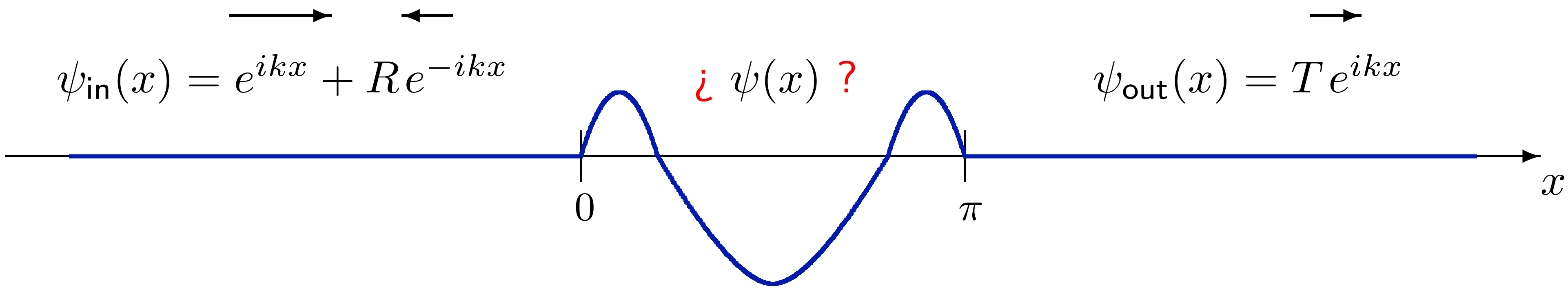

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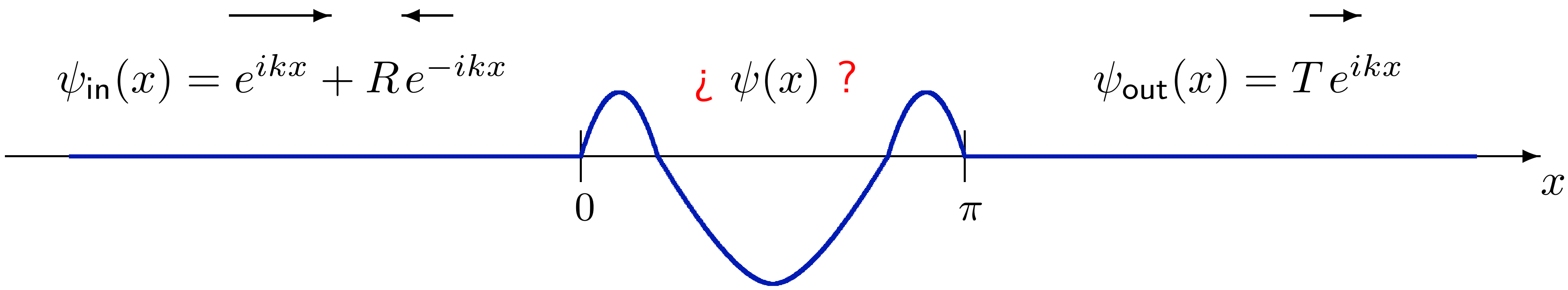

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solutions given by a non-self-adjoint $\mathcal{PT}$-symmetric spectral problem: $\downarrow$

$$\begin{cases} -\psi'' + V\psi = \mu(\alpha)\psi & \text{in} \quad (0,\pi) \\ \psi' + i\alpha\psi = 0 & \text{at} \quad 0,\pi \\ \mu(\alpha) = \alpha^2 & \end{cases}$$

non-Hermitian

The metric operator

[D.K., Siegl, Železný 2011]

Theorem 3. Let $\alpha \notin \mathbb{Z} \setminus \{0\}$.

Then H_α is similar to a self-adjoint operator $h_\alpha := \Omega\, H_\alpha\, \Omega^{-1}$ with the metric

$$\boxed{\Theta := \Omega^* \Omega = I + K}$$

$$\boxed{K(x,y) = \frac{2\,i}{\pi}\, e^{i\frac{\alpha}{2}(x-y)} \sin\left(\frac{\alpha}{2}(x-y)\right) + \frac{i\alpha}{\pi}\left(|y-x| - \pi\right) \operatorname{sgn}(y-x) + \frac{\alpha^2}{\pi}\left(\frac{\pi^2}{4} - xy - \frac{\pi}{2}|y-x|\right)}$$

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non-Hermitian $H_\alpha \longleftrightarrow$ non-local h_α

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Proof. "backward usage of the spectral theorem" [D.K. 2008]

$$E_n = \begin{cases} \alpha^2 & \text{if} \quad n = 0 \\ n^2 = E_n^N = E_n^D & \text{if} \quad n \geq 1 \end{cases} \qquad \phi_n(x) = \begin{cases} \frac{1}{\sqrt{\pi}}\, e^{i\alpha(x+\frac{\pi}{2})} & \text{if} \quad n = 0 \\ \chi_n^N(x) + i\,\frac{\alpha}{n}\,\chi_n^D(x) & \text{if} \quad n \geq 1 \end{cases}$$

$$\Omega := \sum_{n=0}^{\infty} \chi_n^N \langle \phi_n, \cdot\rangle = \chi_0^N\langle\phi_0,\cdot\rangle - \chi_0^N\langle\chi_0^N,\cdot\rangle + \sum_{n=0}^{\infty}\chi_n^N\langle\chi_n^N,\cdot\rangle - i\alpha\sum_{n=1}^{\infty}\frac{1}{n}\,\chi_n^N\langle\chi_n^D,\cdot\rangle$$

$$= \chi_0^N\langle\phi_0,\cdot\rangle - \chi_0^N\langle\chi_0^N,\cdot\rangle + I + \alpha p\,(-\Delta_D)^{-1} \qquad (-\Delta_D = p^*p)$$

q.e.d.

The $\mathcal{C}$ operator

Definition ([Bender, Brody Jones 2002], [Albeverio, Kuzhel 2005]).
Let H be $\mathcal{P}$-self-adjoint and $\mathcal{C}$ bounded.

$$H \text{ is } \boxed{\mathcal{C}\text{-symmetric}} \quad :\Longleftrightarrow \quad \begin{cases} [H, \mathcal{C}] = 0 \\ \mathcal{C}^2 = I \\ \mathcal{P}\mathcal{C} \text{ is a metric for } H \end{cases}$$

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Our explicit result:

$\mathcal{C} = \mathcal{P} + L$ with

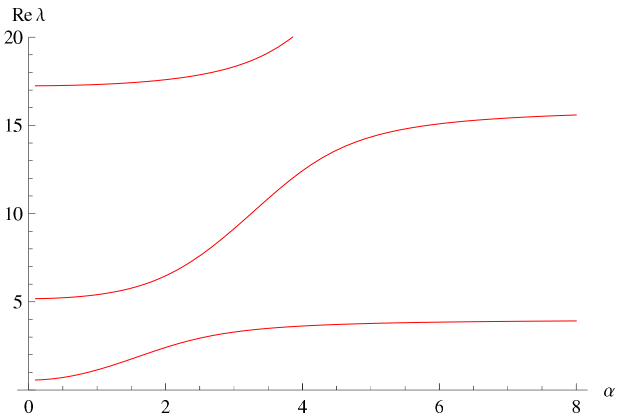
$$\boxed{L(x,y) = \alpha\, e^{-i\alpha(y+x)} \left[\tan(\alpha\tfrac{\pi}{2}) - i\,\mathrm{sgn}(y+x)\right]} \qquad (|\alpha| < 1)$$

General $\mathcal{PT}$-symmetric case

[D.K., Siegl 2010]

$$H_{\alpha,\beta}\psi := -\psi'', \quad D(H_{\alpha,\beta}) := \left\{ \psi \in W^{2,2}(-\tfrac{\pi}{2}, \tfrac{\pi}{2}) \;\middle|\; \psi'(\pm\tfrac{\pi}{2}) + (i\alpha \pm \beta)\,\psi(\pm\tfrac{\pi}{2}) = 0 \right\}$$

$\beta > 0$



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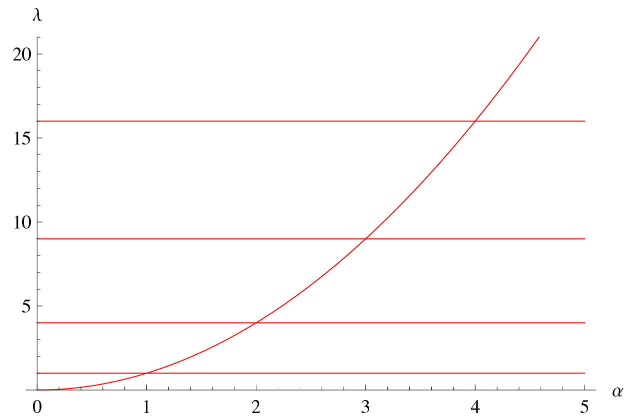

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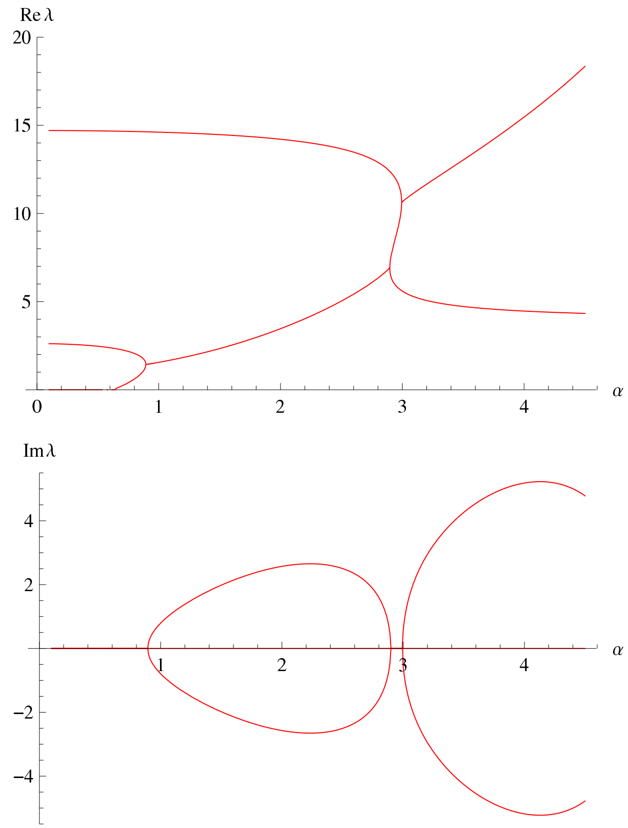

General $\mathcal{PT}$ - symmetric case

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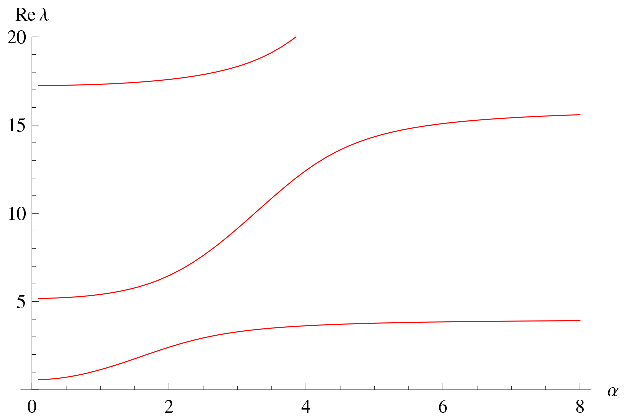

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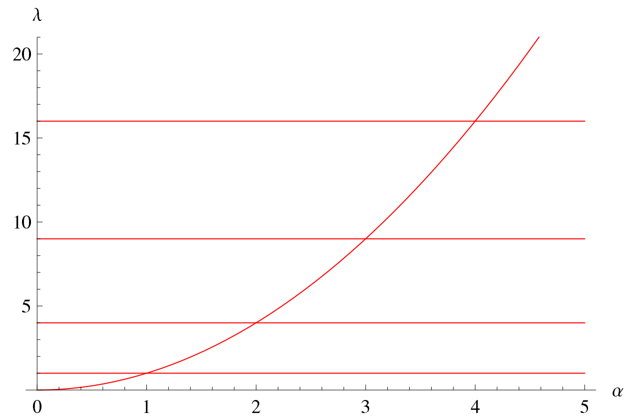

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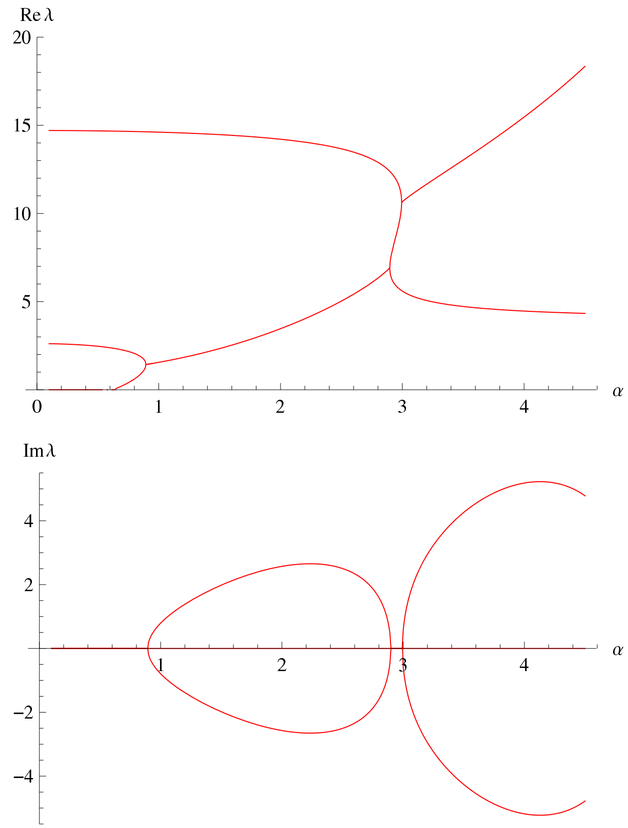

$\Theta = I + K$ with

$$\boxed{K(x,y) = e^{i\alpha(x-y)-\beta|x-y|}\left[c + i\,\alpha\,\mathrm{sgn}(x-y)\right]} \qquad c \in \mathbb{R}$$

$\Theta > 0$ if $\beta > 0$ large or $c^2 + \alpha^2$ small

Imaginary cubic oscillator

[D.K., Siegl 2012]

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ix^3$$ on $L^2(\mathbb{R})$

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- $\left\|(H_h - z)^{-1}\right\| = \mathcal{O}(h^{-n})$, $\forall n > 0$ $\Rightarrow$ contradiction *q.e.d.*

[Davies 1999]

Pseudospectra and $\mathcal{PT}$-symmetry

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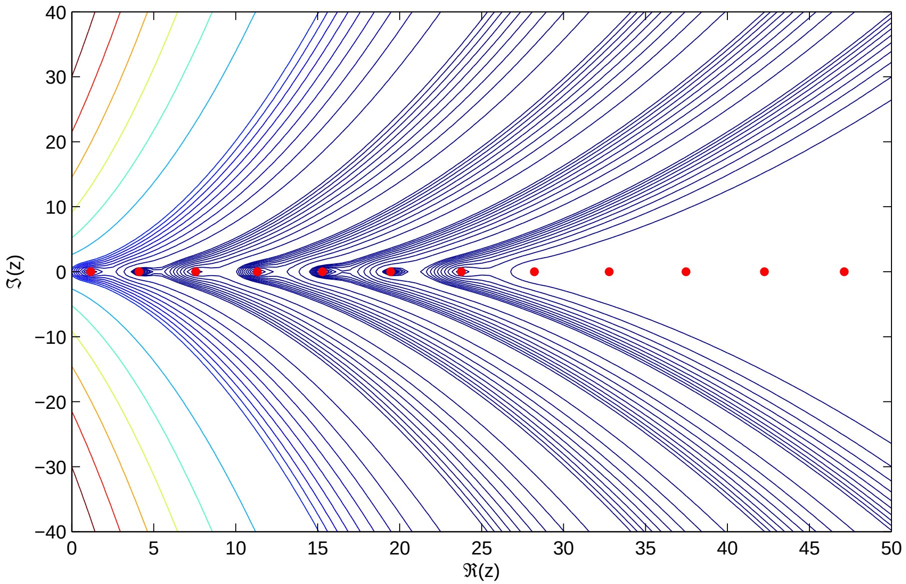
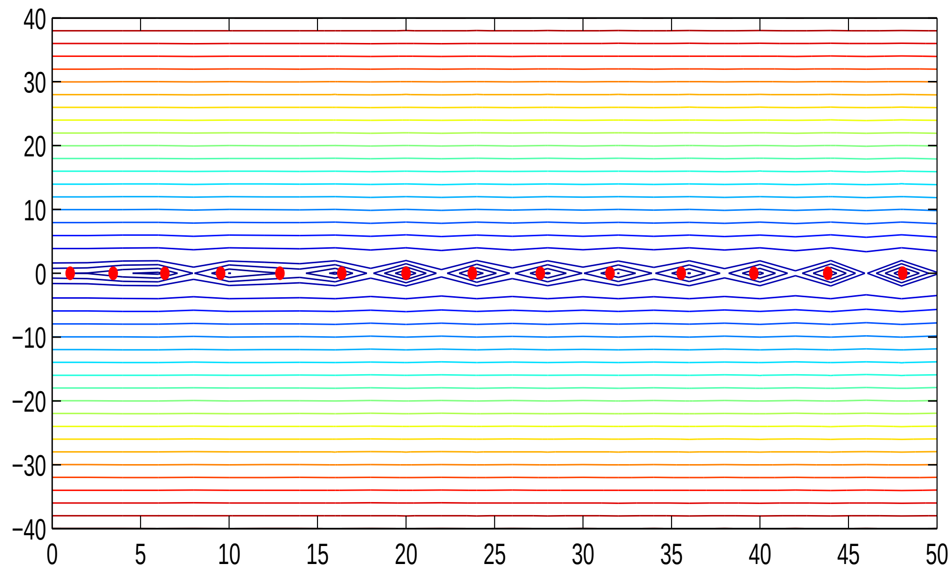
- $\sigma_\varepsilon(H) = \bigcup\limits_{\|V\|<\varepsilon} \sigma(H+V) \quad\Longrightarrow\quad$ spectral instabilities
- $\exists$ metric $\quad\Longrightarrow\quad$ trivial pseudospectrum



ix^3



$|x|^3$

Conclusions

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Ad ix^3 :

¡ metric operator does not exist !

(bad basicity properties, non-trivial pseudospectrum, spectral instabilities)

Collection of open problems

ESF exploratory workshop on

Mathematical aspects of the physics with non-self-adjoint operators

30 August - 3 September 2010
Prague, Czech Republic

`http://www.ujf.cas.cz/ESFxNSA/`

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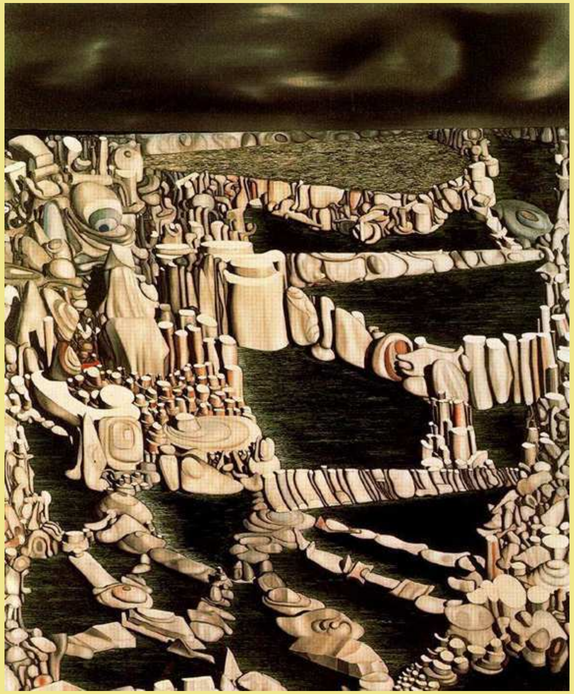

Imaginary numbers 1954 by Y. Tanguy

Studying non-self-adjoint operators is like being a vet rather than a doctor: one has to acquire a much wider range of knowledge, and to accept that one cannot expect to have as high a rate of success when confronted with particular cases.

E. B. Davies 2007

Convenors: J.-P. Gazeau (Paris), D. Krejcirik (Prague), P. Siegl (Prague)

Some of the open problems also available in *Integral Equations Operator Theory*.

My $\mathcal{PT}$-symmetric life

http://gemma.ujf.cas.cz/~david/

- D.K., H. Bíla, M. Znojil: *Closed formula for the metric in the Hilbert space of a $\mathcal{PT}$-symmetric model*; J. Phys. A 39 (2006), 10143–10153.
- D.K.: *Calculation of the metric in the Hilbert space of a $\mathcal{PT}$-symmetric model via the spectral theorem*; J. Phys. A: Math. Theor. 41 (2008) 244012.
- D. Borisov, D.K.: *$\mathcal{PT}$-symmetric waveguides*; Integral Equations Operator Theory 62 (2008), 489–515.
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