Weak and generalized Weyl form of the commutation relation for unbounded operators

Camillo Trapani

Dipartimento di Matematica e Informatica, Università di Palermo, Italy

(Joint work with F.Bagarello and A.Inoue)

PHHQP11, Paris, 2012

Motivations

Motivations

The commutator algebra of operators representing observables plays an important role in QM

Motivations

The commutator algebra of operators representing observables plays an important role in QM

These operators are usually unbounded. This fact poses several problems for a correct mathematical formulation of these notions.

Motivations

The commutator algebra of operators representing observables plays an important role in QM

These operators are usually unbounded. This fact poses several problems for a correct mathematical formulation of these notions.

Let A, B be unbounded operators in Hilbert sp.

- $AB = BA$ can be meaningless
- strong commutation for selfadjoint operators
- Nelson's example
- Weak commutation
- Commutators $[A, B] := AB - BA$ can be meaningless too
- Weyl commutation relations
- Weak form of commutators

When do A, B commute?

Many possible cases:

- A, B bounded operators in $\mathcal{H}$: clear
  - A, B self-adjoint unbounded : strong commutation
    - Commutation of spectral families $\{E_A(\lambda)\}$, $\{E_B(\mu)\}$, $\lambda, \mu \in \mathbb{R}$
    - Commutation of the unitary groups $U_A(t) := e^{iAt}$, $U_B(s) := e^{iBs}$
    - Commutation of resolvent functions $(A - \lambda I)^{-1} \smile (B - \mu I)^{-1}$, $\lambda \in \rho(A), \mu \in \rho(B)$.

Many possible cases:

- A, B bounded operators in $\mathcal{H}$: clear
  - A, B self-adjoint unbounded : strong commutation
    - Commutation of spectral families $\{E_A(\lambda)\}$, $\{E_B(\mu)\}$, $\lambda, \mu \in \mathbb{R}$
    - Commutation of the unitary groups $U_A(t) := e^{iAt}$, $U_B(s) := e^{iBs}$
    - Commutation of resolvent functions $(A - \lambda I)^{-1} \smile (B - \mu I)^{-1}$, $\lambda \in \rho(A), \mu \in \rho(B)$.
  - $A, B \in \mathcal{L}^\dagger(\mathcal{D})$

$$\mathcal{L}^\dagger(\mathcal{D}) := \{\text{closable } A : \; A\mathcal{D} \subseteq \mathcal{D}; \; A^*\mathcal{D} \subseteq \mathcal{D}\}$$

*-algebra $\Rightarrow$ $AB = BA$ well defined.

Many possible cases:

- A, B bounded operators in $\mathcal{H}$: clear
  - A, B self-adjoint unbounded : strong commutation
    - Commutation of spectral families $\{E_A(\lambda)\}$, $\{E_B(\mu)\}$, $\lambda, \mu \in \mathbb{R}$
    - Commutation of the unitary groups $U_A(t) := e^{iAt}$, $U_B(s) := e^{iBs}$
    - Commutation of resolvent functions $(A - \lambda I)^{-1} \smile (B - \mu I)^{-1}$, $\lambda \in \rho(A), \mu \in \rho(B)$.
  - $A, B \in \mathcal{L}^\dagger(\mathcal{D})$

$$\mathcal{L}^\dagger(\mathcal{D}) := \{\text{closable } A : \ A\mathcal{D} \subseteq \mathcal{D};\ A^*\mathcal{D} \subseteq \mathcal{D}\}$$

*-algebra $\Rightarrow AB = BA$ well defined.

But Nelson's example $A, B \in \mathcal{L}^\dagger(\mathcal{D})$
A, B essentially selfadjoint (closures are selfadjoint)
$AB\xi = BA\xi$ for every $\xi \in \mathcal{D}$
but spectral families do not commute!

In hermitian QM strong commutation is a natural concept

- Probabilistic interpretation of the spectral measure:
  $E_A(\cdot)$ spectral family of A
  $\mathrm{Prob}\{(A,\psi) \in \Delta\} = \int_\Delta d\langle E_A(\lambda)\psi|\psi\rangle$

In hermitian QM strong commutation is a natural concept

- Probabilistic interpretation of the spectral measure:
  $E_A(\cdot)$ spectral family of A
  $\text{Prob}\{(A,\psi) \in \Delta\} = \int_\Delta d\langle E_A(\lambda)\psi|\psi\rangle$
- Existence of joint probability distribution:
  If A, B commute strongly
  $\Rightarrow E_A(\cdot)E_B(\cdot)$ spectral measure on the plane
  $E_A(\cdot)E_B(\cdot)$ gives the joint probability distribution (A and B can be measured simultaneously).

In nonhermitian QM ???

$$A, B \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) = \{\text{closable } A : \; D(A) = \mathcal{D}, \; D(A^*) \supset \mathcal{D}\}$$

$A^\dagger := A^* \upharpoonright \mathcal{D}$.

$$A, B \in \mathcal{L}^{\dagger}(\mathcal{D}, \mathcal{H}) = \{\text{closable } A : \; D(A) = \mathcal{D}, \; D(A^*) \supset \mathcal{D}\}$$

$A^{\dagger} := A^* \upharpoonright \mathcal{D}$.

Partial *-algebra w. r. to weak product $\square$: [Antoine, Karwowski]

$A \,\square\, B$ exists iff $\begin{cases} B : \mathcal{D} \to D(A^{\dagger *}) \\ A^{\dagger} : \mathcal{D} \to D(B^*) \end{cases}$

and $(A \,\square\, B)\xi = A^{\dagger *} B\xi, \quad \forall \xi \in \mathcal{D}$.

$$A, B \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) = \{\text{closable } A : \ D(A) = \mathcal{D}, \ D(A^*) \supset \mathcal{D}\}$$

$A^\dagger := A^* \restriction \mathcal{D}$.

Partial *-algebra w. r. to weak product $\square$: [Antoine, Karwowski]

$A \,\square\, B$ exists iff $\begin{cases} B : \mathcal{D} \to D(A^{\dagger *}) \\ A^\dagger : \mathcal{D} \to D(B^*) \end{cases}$

and $(A \,\square\, B)\xi = A^{\dagger *} B\xi, \quad \forall \xi \in \mathcal{D}$.

$A \,\square\, B = B \,\square\, A$: OK algebraically, same problems as before.

Moreover technically complicated.

$$A, B \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) = \{\text{closable } A : \ D(A) = \mathcal{D}, \ D(A^*) \supset \mathcal{D}\}$$

$A^\dagger := A^* \upharpoonright \mathcal{D}$.

Partial *-algebra w. r. to weak product $\square$: [Antoine, Karwowski]

$A \square B$ exists iff $\begin{cases} B : \mathcal{D} \to D(A^{\dagger *}) \\ A^\dagger : \mathcal{D} \to D(B^*) \end{cases}$

and $(A \square B)\xi = A^{\dagger *} B\xi, \quad \forall \xi \in \mathcal{D}$.

$A \square B = B \square A$: OK algebraically, same problems as before.

Moreover technically complicated.

WEAK COMMUTATION: easier to handle

$$\langle B\xi | A^\dagger \eta \rangle = \langle A\xi | B^\dagger \eta \rangle, \quad \forall \xi, \eta \in \mathcal{D}$$

$$A, B \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) = \{\text{closable } A: \ D(A) = \mathcal{D}, \ D(A^*) \supset \mathcal{D}\}$$

$A^\dagger := A^* \upharpoonright \mathcal{D}$.

Partial *-algebra w. r. to weak product $\Box$: [Antoine, Karwowski]

$A \Box B$ exists iff $\begin{cases} B : \mathcal{D} \to D(A^{\dagger *}) \\ A^\dagger : \mathcal{D} \to D(B^*) \end{cases}$

and $(A \Box B)\xi = A^{\dagger *} B\xi, \quad \forall \xi \in \mathcal{D}$.

$A \Box B = B \Box A$: OK algebraically, same problems as before.
Moreover technically complicated.

WEAK COMMUTATION: easier to handle

$$\langle B\xi | A^\dagger \eta \rangle = \langle A\xi | B^\dagger \eta \rangle, \quad \forall \xi, \eta \in \mathcal{D}$$

See J.-P. Antoine, A. Inoue, C. Trapani, Partial *-algebras and their operator realizations, Kluwer 2002.

In $\mathcal{B}(\mathcal{H})$, $[A, B] = AB - BA$ well-defined bounded operator.

The map $(A, B) \to [A, B]$ makes of $B(H)$ a Banach Lie algebra: everything works fine therein!

In $\mathcal{B}(\mathcal{H})$, $[A, B] = AB - BA$ well-defined bounded operator.

The map $(A, B) \to [A, B]$ makes of $B(H)$ a Banach Lie algebra: everything works fine therein!

But representations of Lie algebras involve, in general unbounded operators!

In $\mathcal{B}(\mathcal{H})$, $[A, B] = AB - BA$ well-defined bounded operator.

The map $(A, B) \to [A, B]$ makes of $B(H)$ a Banach Lie algebra: everything works fine therein!

But representations of Lie algebras involve, in general unbounded operators!

In $\mathcal{L}^\dagger(\mathcal{D})$, $[A, B]$ is well-defined, but ...

regularity, i.e. integrability is not guaranteed (Schmüdgen).

In $\mathcal{B}(\mathcal{H})$, $[A, B] = AB - BA$ well-defined bounded operator.

The map $(A, B) \to [A, B]$ makes of $B(H)$ a Banach Lie algebra: everything works fine therein!

But representations of Lie algebras involve, in general unbounded operators!

In $\mathcal{L}^\dagger(\mathcal{D})$, $[A, B]$ is well-defined, but ...

regularity, i.e. integrability is not guaranteed (Schmüdgen).

Even worse in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$!

Heisenberg Lie Algebra $\mathfrak{h}$: generated by three elements $a, b, c \in \mathfrak{h}$ whose Lie brackets are defined by

$$[a, b] = c \quad [a, c] = [b, c] = 0$$

A representation of $\mathfrak{h}$ linear map $\pi : \mathfrak{h} \to$ operator space such that

$$[\pi(a), \pi(b)] = \pi(c) =: \mathbb{1} \;\; \text{identity operator} \; .$$

Theorem

(Wiener, Wielandt, von Neumann) There exists no bounded representation of the Heisenberg algebra.

Domain: $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. Define operators (annihilation , creation)

$$Af = \frac{1}{\sqrt{2}}(xf + D_x f) \quad \text{and } A^\dagger f = \frac{1}{\sqrt{2}}(xf - D_x f)$$

Then $AA^\dagger f - A^\dagger Af = f, \quad \forall f \in \mathcal{S}(\mathbb{R})$

$\sigma(A^\dagger A) = \mathbb{N}$, number operator

Domain: $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. Define operators (annihilation , creation)

$$Af = \frac{1}{\sqrt{2}}(xf + D_x f) \quad \text{and } A^\dagger f = \frac{1}{\sqrt{2}}(xf - D_x f)$$

Then $AA^\dagger f - A^\dagger A f = f, \quad \forall f \in \mathcal{S}(\mathbb{R})$

$\sigma(A^\dagger A) = \mathbb{N}$, number operator

Theorem

(Stone, von Neumann) Any integrable representation π of the Heisenberg Lie algebra is unitarily equivalent to the Schrödinger repr.

Integrability: $\exists$ connected and simply connected Lie group G and a unitary representation U of G such that $\pi = dU$.

Problem

Study of operators A, B such that $B \neq A^\dagger$ and $[A, B] = \mathbb{1}$ in some sense (Bagarello, Inoue, CT 2011, 2012)

- Bagarello's pseudo-bosons
- nonintegrable repr. of CCR.

A, B closed operators, dense domains $D(A)$ and $D(B)$ in $\mathcal{H}$.

To give a meaning to $AB - BA = \mathbb{1}$ we suppose $\exists$ a dense subspace $\mathcal{D}$ of $\mathcal{H}$ such that

(D.1) $\mathcal{D} \subset D(AB) \cap D(BA)$ $[D(AB) = \{\xi \in D(B) : B\xi \in D(A)\}]$.

(D.2) $AB\xi - BA\xi = \xi, \quad \forall \xi \in \mathcal{D}$.

(D.3) $\mathcal{D} \subset D(A^*) \cap D(B^*)$.

Then $S := A \upharpoonright \mathcal{D}$ and $T := B \upharpoonright D$ belong to (partial *-algebra) $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ and satisfy

$$\langle T\xi | S^\dagger \eta\rangle - \langle S\xi | T^\dagger \eta\rangle = \langle \xi | \eta\rangle, \quad \forall \xi, \eta \in \mathcal{D}.$$

The study of this commutation relation is our main matter.

Particular cases: S and/or T are generators of some weakly continuous semigroup $V(t)$ of bdd operators.

Then $S := A \restriction \mathcal{D}$ and $T := B \restriction D$ belong to (partial *-algebra) $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ and satisfy

$$\langle T\xi | S^\dagger \eta\rangle - \langle S\xi | T^\dagger \eta\rangle = \langle \xi|\eta\rangle, \quad \forall \xi, \eta \in \mathcal{D}.$$

The study of this commutation relation is our main matter.

Particular cases: S and/or T are generators of some weakly continuous semigroup $V(t)$ of bdd operators.

$X_0 \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is the $\mathcal{D}$-generator of $V(t)$ if

$$\lim_{t\to 0}\langle \frac{V(t) - \mathbb{1}}{t}\xi|\eta\rangle = \langle X_0\xi|\eta\rangle, \ \forall \xi, \eta \in \mathcal{D}.$$

Definition

Let $S, T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. *We say that the c. r.* $[S, T] = \mathbb{1}_{\mathcal{D}}$ *holds*

Definition

Let $S, T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. *We say that the c. r.* $[S, T] = \mathbb{1}_\mathcal{D}$ *holds*

(CR.1) *in* $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ *if,* $S \,\square\, T$ *well-defined* $\Rightarrow$ $T \,\square\, S$ *well-defined too and* $S \,\square\, T - T \,\square\, S = \mathbb{1}_\mathcal{D}$;

Definition

Let $S, T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$. We say that the c. r. $[S, T] = \mathbb{1}_\mathcal{D}$ holds

(CR.1) *in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ if, $S \,\square\, T$ well-defined $\Rightarrow$ $T \,\square\, S$ well-defined too and $S \,\square\, T - T \,\square\, S = \mathbb{1}_\mathcal{D}$;*

(CR.2) *in weak sense if*

$$\langle T\xi | S^\dagger \eta\rangle - \langle S\xi | T^\dagger \eta\rangle = \langle \xi | \eta\rangle, \quad \forall \xi, \eta \in \mathcal{D};$$

Definition

Let $S, T \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. We say that the c. r. $[S,T]=1\!\!1_{\mathcal{D}}$ holds

(CR.1) *in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ if, $S \,\square\, T$ well-defined $\Rightarrow$ $T \,\square\, S$ well-defined too and $S \,\square\, T - T \,\square\, S = 1\!\!1_{\mathcal{D}}$;*

(CR.2) *in weak sense if*

$$\langle T\xi|S^\dagger\eta\rangle - \langle S\xi|T^\dagger\eta\rangle = \langle\xi|\eta\rangle, \quad \forall\xi,\eta\in\mathcal{D};$$

(CR.3) *in quasi-strong sense if S is the $\mathcal{D}$-generator of a w-continuous semigr. of bdd operators $V_S(\alpha)$ and*

$$\langle V_S(\alpha)T\xi|\eta\rangle - \langle V_S(\alpha)\xi|T^\dagger\eta\rangle = \alpha\langle V_S(\alpha)\xi|\eta\rangle, \quad \forall\xi,\eta\in\mathcal{D}, \forall\alpha\geqslant 0;$$

Definition

Let $S, T \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. We say that the c. r. $[S,T] = \mathbb{1}_\mathcal{D}$ holds

(CR.1) *in $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ if, $S \,\Box\, T$ well-defined $\Rightarrow$ $T \,\Box\, S$ well-defined too and $S \,\Box\, T - T \,\Box\, S = \mathbb{1}_\mathcal{D}$;*

(CR.2) *in weak sense if*

$$\langle T\xi | S^\dagger \eta\rangle - \langle S\xi | T^\dagger \eta\rangle = \langle \xi|\eta\rangle, \quad \forall \xi,\eta \in \mathcal{D};$$

(CR.3) *in quasi-strong sense if S is the $\mathcal{D}$-generator of a w-continuous semigr. of bdd operators $V_S(\alpha)$ and*

$$\langle V_S(\alpha) T\xi|\eta\rangle - \langle V_S(\alpha)\xi | T^\dagger\eta\rangle = \alpha\langle V_S(\alpha)\xi|\eta\rangle, \quad \forall \xi,\eta\in\mathcal{D}, \forall\alpha \geqslant 0;$$

(CR.4) *in strong sense if S and T are $\mathcal{D}$-generators of w-continuous semigr. of bdd operators $V_S(\alpha), V_T(\beta)$ satisfying the* generalized *Weyl c.r.*

$$V_S(\alpha)V_T(\beta) = e^{\alpha\beta} V_T(\beta)V_S(\alpha), \quad \forall \alpha,\beta \geqslant 0.$$

$$(\mathrm{CR}.4) \Rightarrow (\mathrm{CR}.3) \Rightarrow (\mathrm{CR}.2) \Rightarrow (\mathrm{CR}.1).$$

$(\mathrm{CR}.4) \Rightarrow (\mathrm{CR}.3) \Rightarrow (\mathrm{CR}.2) \Rightarrow (\mathrm{CR}.1)$.

Implications in the other direction: FALSE:

(CR.4) $\Rightarrow$ (CR.3) $\Rightarrow$ (CR.2) $\Rightarrow$ (CR.1).

Implications in the other direction: FALSE:

Example

$\exists$ two essentially selfadjoint operators P, Q with common invariant dense domain $\mathcal{D}$ such that $PQ\xi - QP\xi = -i\xi$, for $\xi \in \mathcal{D}$, but the unitary groups $U_P(t), U_Q(s)$ generated by $\overline{P}$, $\overline{Q}$ do not satisfy the Weyl commutation relation $U_P(t)U_Q(s) = e^{its}U_Q(s)U_P(t)$, $s, t \in \mathbb{R}$. (Fulgede, Schmüdgen).

Existence of eigenvectors

Parallel to the case $[A, A^\dagger] = \mathbb{1}$ when a *vacuum* ξ_0 exists: $A\xi_0 = 0$

$S, T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, satisfy $[S, T] = \mathbb{1}_{\mathcal{D}}$ in weak sense.

Existence of eigenvectors

Parallel to the case $[A, A^\dagger] = \mathbb{1}$ when a *vacuum* ξ_0 exists: $A\xi_0 = 0$

$S, T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, satisfy $[S, T] = \mathbb{1}_{\mathcal{D}}$ in weak sense.
Assume:

- $\exists$ a vector $0 \neq \xi_0 \in \mathcal{D}$ such that $S\xi_0 = 0$.
- $T^k\xi_0 \in \mathcal{D}$, $k \leqslant n$

Existence of eigenvectors

Parallel to the case $[A, A^\dagger] = \mathbb{1}$ when a *vacuum* ξ_0 exists: $A\xi_0 = 0$

$S, T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, satisfy $[S, T] = \mathbb{1}_\mathcal{D}$ in weak sense.
Assume:

- $\exists$ a vector $0 \neq \xi_0 \in \mathcal{D}$ such that $S\xi_0 = 0$.
- $T^k\xi_0 \in \mathcal{D}$, $k \leqslant n$

Consider the operators

$$N := TS^{\dagger *}, \qquad N^\sharp : S^{\dagger *}T.$$

N acts as number operator on a subspace $\mathcal{N}_0$ of $\mathcal{H}$.

Existence of eigenvectors

Parallel to the case $[A, A^\dagger] = \mathbb{1}$ when a *vacuum* ξ_0 exists: $A\xi_0 = 0$

$S, T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, satisfy $[S, T] = \mathbb{1}_\mathcal{D}$ in weak sense.
Assume:
- $\exists$ a vector $0 \neq \xi_0 \in \mathcal{D}$ such that $S\xi_0 = 0$.
- $T^k\xi_0 \in \mathcal{D}$, $k \leqslant n$

Consider the operators

$$N := TS^{\dagger *}, \qquad N^\sharp : S^{\dagger *}T.$$

N acts as number operator on a subspace $\mathcal{N}_0$ of $\mathcal{H}$.

(i) $T^n\xi_0$ is an eigenvector of $N = TS^{\dagger *}$ with eigenvalue n;

Existence of eigenvectors

Parallel to the case $[A, A^\dagger] = \mathbb{1}$ when a *vacuum* ξ_0 exists: $A\xi_0 = 0$

$S, T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$, satisfy $[S, T] = \mathbb{1}_{\mathcal{D}}$ in weak sense.
Assume:

- $\exists$ a vector $0 \neq \xi_0 \in \mathcal{D}$ such that $S\xi_0 = 0$.
- $T^k\xi_0 \in \mathcal{D}$, $k \leqslant n$

Consider the operators

$$N := TS^{\dagger *}, \qquad N^\sharp : S^{\dagger *}T.$$

N acts as number operator on a subspace $\mathcal{N}_0$ of $\mathcal{H}$.

(i) $T^n\xi_0$ is an eigenvector of $N = TS^{\dagger *}$ with eigenvalue n;

(ii) $T^{n-1}\xi_0$ is eigenvector of $N^\sharp = S^{\dagger *}T$ with eigenvalue n.

- The largest n for which $T^n\xi_0 \in \mathcal{D}$ may be finite or infinite.
  $\mathcal{N}_0 := \mathrm{lspan}\{\xi_0, T\xi_0, \dots T^n\xi_0\}$.
  $N := TS^{\dagger *}$ leaves $\mathcal{N}_0$ invariant.

- The largest n for which $T^n\xi_0 \in \mathcal{D}$ may be finite or infinite.
  $\mathcal{N}_0 := \mathrm{lspan}\{\xi_0, T\xi_0, \dots T^n\xi_0\}$.
  $N := TS^{\dagger *}$ leaves $\mathcal{N}_0$ invariant.

- point spectrum: $\sigma_p(N_0) = \{0, 1, \dots, n\}$ in $\mathcal{N}_0$, $n \in \mathbb{N} \cup \{\infty\} =$ largest natural number s.t. $T\xi_0, T^2\xi_0, \dots T^{n-1}\xi_0$ all belong to $\mathcal{D}$.
  Each eigenvalue is simple (in $\mathcal{N}_0$).

- The largest n for which $T^n\xi_0 \in \mathcal{D}$ may be finite or infinite.
  $\mathcal{N}_0 := \text{lspan}\{\xi_0, T\xi_0, \dots T^n\xi_0\}$.
  $N := TS^{\dagger *}$ leaves $\mathcal{N}_0$ invariant.

- point spectrum: $\sigma_p(N_0) = \{0, 1, \dots, n\}$ in $\mathcal{N}_0$, $n \in \mathbb{N} \cup \{\infty\} =$ largest natural number s.t. $T\xi_0, T^2\xi_0, \dots T^{n-1}\xi_0$ all belong to $\mathcal{D}$.
  Each eigenvalue is simple (in $\mathcal{N}_0$).

- $ST^k\xi - (T^{\dagger *})^k S\xi = kT^{k-1}\xi, \quad k \leqslant n.$

Examples

Hilbert space $L^2(\mathbb{R}, wdx)$; the weight $w \in C^1(\mathbb{R})$, $\quad w > 0$, s.t.

- $\lim_{|x|\to+\infty} w(x) = 0$;
- $\int_{\mathbb{R}} w(x)dx < \infty$.

Examples

Hilbert space $L^2(\mathbb{R}, wdx)$; the weight $w \in C^1(\mathbb{R})$, $\quad w > 0$, s.t.

- $\lim_{|x| \to +\infty} w(x) = 0$;
- $\int_{\mathbb{R}} w(x)dx < \infty$.

$$D(\mathrm{p}) = \left\{ f \in L^2(\mathbb{R}, wdx) : \exists g \in L^2(\mathbb{R}, wdx), f(x) = \int_{-\infty}^{x} g(t)dt \right\}.$$

Shortly, $f'(x) := g(x)$, for $f \in D(\mathrm{p})$.

Examples

Hilbert space $L^2(\mathbb{R}, wdx)$; the weight $w \in C^1(\mathbb{R})$, $\quad w > 0$, s.t.

- $\lim_{|x| \to +\infty} w(x) = 0$;
- $\int_{\mathbb{R}} w(x)dx < \infty$.

$$D(\mathrm{p}) = \left\{ f \in L^2(\mathbb{R}, wdx) : \exists g \in L^2(\mathbb{R}, wdx), f(x) = \int_{-\infty}^{x} g(t)dt \right\}.$$

Shortly, $f'(x) := g(x)$, for $f \in D(\mathrm{p})$.

$$D(\mathrm{q}) = \{ f \in L^2(\mathbb{R}, wdx) : xf(x) \in L^2(\mathbb{R}, wdx) \}.$$

Examples

Hilbert space $L^2(\mathbb{R}, wdx)$; the weight $w \in C^1(\mathbb{R})$, $w > 0$, s.t.

- $\lim_{|x|\to+\infty} w(x) = 0$;
- $\int_{\mathbb{R}} w(x)dx < \infty$.

$$D(\mathrm{p}) = \left\{ f \in L^2(\mathbb{R}, wdx) : \exists g \in L^2(\mathbb{R}, wdx), f(x) = \int_{-\infty}^{x} g(t)dt \right\}.$$

Shortly, $f'(x) := g(x)$, for $f \in D(\mathrm{p})$.

$$D(\mathrm{q}) = \{ f \in L^2(\mathbb{R}, wdx) : xf(x) \in L^2(\mathbb{R}, wdx)\}.$$

$\mathcal{D} := D(\mathrm{q}) \cap D(\mathrm{p})$. Define

$$(Sf)(x) = f'(x), \qquad (Tf)(x) = xf(x), \quad f \in \mathcal{D}$$

Both map $\mathcal{D}$ into $L^2(\mathbb{R}, wdx)$. T is symmetric in $\mathcal{D}$.
Formally

$$(S^* g)(x) = -g'(x) - g(x)\frac{w'(x)}{w(x)}.$$

If $w'/w \in L^\infty(\mathbb{R})$, $S \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ and $[S, T] = \mathbb{1}_{\mathcal{D}}$ (weak sense).

Make some particular choices for w

$\boxed{w(x) = w_\alpha(x) = (1 + x^4)^{-\alpha},\ \alpha > \frac{3}{4}}$ $u_0(x) = 1$, is in $L^2(\mathbb{R}, w_\alpha dx)$ for $\alpha > \frac{3}{4}$. It satisfies $Su_0 = 0$

The largest n for which $T^n u_0$ belongs to $\mathcal{D}$ satisfies $n < 2\alpha - \frac{3}{2}$. $\Rightarrow$ $\dim \mathcal{N}_0$ is $\left[2\alpha - \frac{3}{2}\right] + 1$.

$\boxed{w(x) = w_\alpha(x) = (1+x^4)^{-\alpha},\ \alpha > \frac{3}{4}}$ $u_0(x) = 1$, is in $L^2(\mathbb{R}, w_\alpha dx)$ for $\alpha > \frac{3}{4}$. It satisfies $Su_0 = 0$

The largest n for which $T^n u_0$ belongs to $\mathcal{D}$ satisfies $n < 2\alpha - \frac{3}{2}$. $\Rightarrow$ $\dim \mathcal{N}_0$ is $\left[2\alpha - \frac{3}{2}\right] + 1$.

$\boxed{w(x) = e^{-x^2/2}}$ $\mathcal{D} =$ all polynomials in x.

The functions $u_k(x) = x^k$, $k = 1, 2, \ldots$, belong to $\mathcal{D}$ and $TS^{\dagger *} u_k = k u_k$ for every $k \in \mathbb{N}$.

The subspace $\mathcal{N}_0$ coincides in this case with $\mathcal{D}$.

Every complex number λ with $\Re\lambda > -\frac{1}{2}$ is an eigenvalue of $N = TS^{\dagger *}$; but the corresponding eigenvector is in $\mathcal{D}$ if and only if $\Re\lambda$ is a natural number.

Remarks

The subspace $\mathcal{N}_0$ spanned by $\{T^k\xi_0, k \in \mathbb{N}\}$ can be finite dimensional.

Remarks

The subspace $\mathcal{N}_0$ spanned by $\{T^k\xi_0, k \in \mathbb{N}\}$ can be finite dimensional.

Thus $N := (TS^{\dagger *})_0$ is a bounded symmetric operator on $\mathcal{N}_0 \cong \mathbb{C}^n$, having the numbers $0, 1, \ldots, n$ as eigenvalues.

Remarks

The subspace $\mathcal{N}_0$ spanned by $\{T^k\xi_0, k \in \mathbb{N}\}$ can be finite dimensional.

Thus $N := (TS^{\dagger *})_0$ is a bounded symmetric operator on $\mathcal{N}_0 \cong \mathbb{C}^n$, having the numbers $0, 1, \ldots, n$ as eigenvalues.

Hence N is positive and thus $\exists$ an operator $C \in \mathcal{B}(\mathcal{N})$ such that $N = C^\dagger C$. None of the possible solutions of this operator equation can, however, satisfy the commutation relation $[C, C^\dagger] = \mathbb{1}$, due to Wiener -Wielandt - von Neumann theorem.

Remarks

The subspace $\mathcal{N}_0$ spanned by $\{T^k \xi_0, k \in \mathbb{N}\}$ can be finite dimensional.

Thus $N := (TS^{\dagger *})_0$ is a bounded symmetric operator on $\mathcal{N}_0 \cong \mathbb{C}^n$, having the numbers $0, 1, \ldots, n$ as eigenvalues.

Hence N is positive and thus $\exists$ an operator $C \in \mathcal{B}(\mathcal{N})$ such that $N = C^\dagger C$. None of the possible solutions of this operator equation can, however, satisfy the commutation relation $[C, C^\dagger] = \mathbb{1}$, due to Wiener -Wielandt - von Neumann theorem.

If $\mathcal{N}_0$ is infinite dimensional then N may fail to be symmetric, as the last example shows.

Intertwining operators

$[S, T] = 1\!\!1_{\mathcal{D}}$ (weak sense) $\Rightarrow [T^\dagger, S^\dagger] = 1\!\!1_{\mathcal{D}}$ (weak sense).

Assume $\exists$ also $0 \neq \eta_0 \in \mathcal{D}$ s. t. $T^\dagger \eta_0 = 0$
$S^\dagger \eta_0$, $(S^\dagger)^2 \eta_0, \ldots (S^\dagger)^{m-1} \eta_0$ all belong to $\mathcal{D}$.

$[S, T] = \mathbb{1}_{\mathcal{D}}$ (weak sense) $\Rightarrow [T^\dagger, S^\dagger] = \mathbb{1}_{\mathcal{D}}$ (weak sense).

Assume $\exists$ also $0 \neq \eta_0 \in \mathcal{D}$ s. t. $T^\dagger \eta_0 = 0$
$S^\dagger \eta_0$, $(S^\dagger)^2 \eta_0, \dots$ $(S^\dagger)^{m-1}\eta_0$ all belong to $\mathcal{D}$. Consider

$$M := S^\dagger T^*, \qquad M^\sharp := T^* S^\dagger.$$

M is a number operator on some subspace $\mathcal{M}_0$

$[S,T] = \mathbb{1}_{\mathcal{D}}$ (weak sense) $\Rightarrow [T^\dagger, S^\dagger] = \mathbb{1}_{\mathcal{D}}$ (weak sense).

Assume $\exists$ also $0 \neq \eta_0 \in \mathcal{D}$ s. t. $T^\dagger \eta_0 = 0$
$S^\dagger \eta_0$, $(S^\dagger)^2 \eta_0, \dots$ $(S^\dagger)^{m-1}\eta_0$ all belong to $\mathcal{D}$. Consider

$$M := S^\dagger T^*, \qquad M^\sharp := T^* S^\dagger.$$

M is a number operator on some subspace $\mathcal{M}_0$

$m \in \mathbb{N} \cup \{\infty\} :=$ largest number satisfying assumptions

Intertwining operators

$[S, T] = \mathbb{1}_{\mathcal{D}}$ (weak sense) $\Rightarrow [T^\dagger, S^\dagger] = \mathbb{1}_{\mathcal{D}}$ (weak sense).

Assume $\exists$ also $0 \neq \eta_0 \in \mathcal{D}$ s. t. $T^\dagger \eta_0 = 0$
$S^\dagger \eta_0$, $(S^\dagger)^2 \eta_0, \dots$ $(S^\dagger)^{m-1} \eta_0$ all belong to $\mathcal{D}$. Consider

$$M := S^\dagger T^*, \qquad M^\sharp := T^* S^\dagger.$$

M is a number operator on some subspace $\mathcal{M}_0$

$m \in \mathbb{N} \cup \{\infty\} :=$ largest number satisfying assumptions

$\sigma_p(M) = \{0, 1, \dots, m\}$

Intertwining operators

$[S, T] = \mathbb{1}_{\mathcal{D}}$ (weak sense) $\Rightarrow [T^\dagger, S^\dagger] = \mathbb{1}_{\mathcal{D}}$ (weak sense).

Assume $\exists$ also $0 \neq \eta_0 \in \mathcal{D}$ s. t. $T^\dagger \eta_0 = 0$
$S^\dagger \eta_0$, $(S^\dagger)^2 \eta_0, \ldots$ $(S^\dagger)^{m-1} \eta_0$ all belong to $\mathcal{D}$. Consider

$$M := S^\dagger T^*, \qquad M^\sharp := T^* S^\dagger.$$

M is a number operator on some subspace $\mathcal{M}_0$

$m \in \mathbb{N} \cup \{\infty\} :=$ largest number satisfying assumptions

$\sigma_p(M) = \{0, 1, \ldots, m\}$

Any relation between n and m? No, in general.

Indeed, the operators S, T considered in the second case of the Example one finds $n = \infty$ and $m = 0$.

Assume $n = m = \infty$.

$\xi_k := \frac{1}{\sqrt{k!}} T^k \xi_0,\ k = 1, \ldots, n \quad \eta_r := \frac{1}{\sqrt{r!}} (S^\dagger)^r \eta_0,\ r = 1, \ldots, m.$

Choose normalization of ξ_0 and η_0 s.t. $\langle \xi_0 | \eta_0 \rangle = 1$.

$\mathcal{F}_\xi := \{\xi_k\}$ and $\mathcal{F}_\eta := \{\eta_r\}$ are biorthogonal: $\langle \xi_i | \eta_j \rangle = \delta_{i,j}, \quad \forall i, j \in \mathbb{N}$.

Assume $n = m = \infty$.

$\xi_k := \frac{1}{\sqrt{k!}} T^k \xi_0,\ k = 1, \ldots, n \quad \eta_r := \frac{1}{\sqrt{r!}} (S^\dagger)^r \eta_0,\ r = 1, \ldots, m.$

Choose normalization of ξ_0 and η_0 s.t. $\langle \xi_0 | \eta_0 \rangle = 1$.

$\mathcal{F}_\xi := \{\xi_k\}$ and $\mathcal{F}_\eta := \{\eta_r\}$ are biorthogonal: $\langle \xi_i | \eta_j \rangle = \delta_{i,j}, \quad \forall i, j \in \mathbb{N}$.

Define intertwining operators:

$K_\xi(\eta_j) = \xi_j,\ j \in \mathbb{N} \qquad K_\eta(\xi_j) = \eta_j,\ j \in \mathbb{N}.$

$K_\eta = K_\xi^{-1}$, both unbounded in general; they obey intertwining relations:

$$K_\eta\, N\phi = M\, K_\eta \phi,\ \forall \phi \in \mathcal{M}_0;$$

$$K_\xi\, M\psi = N\, K_\xi \psi,\ \forall \psi \in \mathcal{N}_0.$$

Assume $n = m = \infty$.

$\xi_k := \frac{1}{\sqrt{k!}} T^k \xi_0$, $k = 1, \ldots, n$ $\quad \eta_r := \frac{1}{\sqrt{r!}} (S^\dagger)^r \eta_0$, $r = 1, \ldots, m$.

Choose normalization of ξ_0 and η_0 s.t. $\langle \xi_0 | \eta_0 \rangle = 1$.

$\mathcal{F}_\xi := \{\xi_k\}$ and $\mathcal{F}_\eta := \{\eta_r\}$ are biorthogonal: $\langle \xi_i | \eta_j \rangle = \delta_{i,j}$, $\quad \forall i, j \in \mathbb{N}$.

Define intertwining operators:

$K_\xi(\eta_j) = \xi_j$, $j \in \mathbb{N}$ $\quad K_\eta(\xi_j) = \eta_j$, $j \in \mathbb{N}$.

$K_\eta = K_\xi^{-1}$, both unbounded in general; they obey intertwining relations:

$$K_\eta \, N\phi = M \, K_\eta \phi, \ \forall \phi \in \mathcal{M}_0;$$

$$K_\xi \, M\psi = N \, K_\xi \psi, \ \forall \psi \in \mathcal{N}_0.$$

If $\mathcal{N}_0 = \mathcal{M}_0 = \mathcal{H}$ and K_ξ, K_η bounded, then $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$ are Riesz bases of $\mathcal{H}$: $\exists \, c, C > 0$ such that

$$c \sum_j |a_j|^2 \leqslant \left\| \sum_j a_j \xi_j \right\|^2 \leqslant C \sum_j |a_j|^2, \ \forall \{a_n\} \in \ell^2$$

and they span $\mathcal{H}$.

Assume $n = m = \infty$.

$\xi_k := \frac{1}{\sqrt{k!}} T^k \xi_0,\ k = 1, \ldots, n \quad \eta_r := \frac{1}{\sqrt{r!}} (S^\dagger)^r \eta_0,\ r = 1, \ldots, m.$

Choose normalization of ξ_0 and η_0 s.t. $\langle \xi_0 | \eta_0 \rangle = 1$.

$\mathcal{F}_\xi := \{\xi_k\}$ and $\mathcal{F}_\eta := \{\eta_r\}$ are biorthogonal: $\langle \xi_i | \eta_j \rangle = \delta_{i,j}, \quad \forall i, j \in \mathbb{N}$.

Define intertwining operators:

$K_\xi(\eta_j) = \xi_j,\ j \in \mathbb{N} \quad K_\eta(\xi_j) = \eta_j,\ j \in \mathbb{N}$.

$K_\eta = K_\xi^{-1}$, both unbounded in general; they obey intertwining relations:

$$K_\eta\, N\phi = M\, K_\eta \phi,\ \forall \phi \in \mathcal{M}_0;$$

$$K_\xi\, M\psi = N\, K_\xi \psi,\ \forall \psi \in \mathcal{N}_0.$$

If $\mathcal{N}_0 = \mathcal{M}_0 = \mathcal{H}$ and K_ξ, K_η bounded, then $\mathcal{F}_\xi$ and $\mathcal{F}_\eta$ are Riesz bases of $\mathcal{H}$: $\exists\, c, C > 0$ such that

$$c \sum_j |a_j|^2 \leqslant \left\| \sum_j a_j \xi_j \right\|^2 \leqslant C \sum_j |a_j|^2,\ \forall \{a_n\} \in \ell^2$$

and they span $\mathcal{H}$.

An orthonormal basis $\mathcal{E} = \{e_j\}$ can be defined by, for instance, $e_j = K_\eta^{1/2} \xi_j$.

Some consequences of (CR.3)

Assume $[S, T] = \mathbb{1}_{\mathcal{D}}$ in quasi-strong sense; i.e.

$$\langle V_S(\alpha) T\xi|\eta\rangle - \langle V_S(\alpha)\xi|T^\dagger\eta\rangle = \alpha\langle V_S(\alpha)\xi|\eta\rangle, \quad \forall \xi, \eta \in \mathcal{D}; \alpha \geqslant 0.$$

Apply the Cauchy-Schwarz inequality ($\xi = \eta$) $\Rightarrow \forall z \in \mathbb{C}, \alpha \geqslant 0$,

$$\alpha|\langle V_S(\alpha)\xi|\xi\rangle| \leqslant 2\max\{\|(T - z)\xi\|, \|(T^\dagger - \bar{z})\xi\|\} \max\{\|V_S(\alpha)\xi\|, \|V_S(\alpha)^*\xi\|\}. \quad (1)$$

Some consequences of (CR.3)

Assume $[S, T] = \mathbb{1}_{\mathcal{D}}$ in quasi-strong sense; i.e.

$$\langle V_S(\alpha)T\xi|\eta\rangle - \langle V_S(\alpha)\xi|T^\dagger\eta\rangle = \alpha\langle V_S(\alpha)\xi|\eta\rangle, \quad \forall\xi,\eta \in \mathcal{D}; \alpha \geqslant 0.$$

Apply the Cauchy-Schwarz inequality $(\xi = \eta) \Rightarrow \forall z \in \mathbb{C}, \alpha \geqslant 0$,

$$\alpha|\langle V_S(\alpha)\xi|\xi\rangle| \leqslant 2\max\{\|(T-z)\xi\|, \|(T^\dagger - \bar{z})\xi\|\}\max\{\|V_S(\alpha)\xi\|, \|V_S(\alpha)^*\xi\|\}. \quad (1)$$

Consequences:

Some consequences of (CR.3)

Assume $[S, T] = \mathbb{1}_{\mathcal{D}}$ in quasi-strong sense; i.e.

$$\langle V_S(\alpha) T\xi|\eta\rangle - \langle V_S(\alpha)\xi|T^{\dagger}\eta\rangle = \alpha\langle V_S(\alpha)\xi|\eta\rangle, \quad \forall\xi,\eta \in \mathcal{D}; \alpha \geqslant 0.$$

Apply the Cauchy-Schwarz inequality $(\xi = \eta) \Rightarrow \forall z \in \mathbb{C}, \alpha \geqslant 0$,

$$\alpha|\langle V_S(\alpha)\xi|\xi\rangle| \leqslant 2\max\{\|(T-z)\xi\|, \|(T^{\dagger}-\bar{z})\xi\|\}\max\{\|V_S(\alpha)\xi\|, \|V_S(\alpha)^*\xi\|\}. \quad (1)$$

Consequences:

- if $T = T^{\dagger} \Rightarrow \sigma_p(T) = \emptyset$.

Some consequences of (CR.3)

Assume $[S, T] = \mathbb{1}_{\mathcal{D}}$ in quasi-strong sense; i.e.

$$\langle V_S(\alpha) T\xi|\eta\rangle - \langle V_S(\alpha)\xi|T^\dagger\eta\rangle = \alpha\langle V_S(\alpha)\xi|\eta\rangle, \quad \forall \xi, \eta \in \mathcal{D}; \alpha \geqslant 0.$$

Apply the Cauchy-Schwarz inequality ($\xi = \eta$) $\Rightarrow \forall z \in \mathbb{C}, \alpha \geqslant 0$,

$$\alpha|\langle V_S(\alpha)\xi|\xi\rangle| \leqslant 2\max\{\|(T - z)\xi\|, \|(T^\dagger - \bar{z})\xi\|\} \max\{\|V_S(\alpha)\xi\|, \|V_S(\alpha)^*\xi\|\}. \quad (1)$$

Consequences:

- if $T = T^\dagger \Rightarrow \sigma_p(T) = \emptyset$.
- If V_S uniformly bounded, i.e. $\|V_S(\alpha)\| \leqslant M$, $\forall \alpha \geqslant 0$, (e.g. V_S isometries or contractions), by (1) $\lim_{\alpha\to\infty} |\langle V_S(\alpha)\xi|\xi\rangle| = 0, \quad \forall \xi \in \mathcal{H}$.

Some consequences of (CR.3)

Assume $[S, T] = \mathbb{1}_{\mathcal{D}}$ in quasi-strong sense; i.e.

$$\langle V_S(\alpha) T\xi|\eta\rangle - \langle V_S(\alpha)\xi|T^\dagger\eta\rangle = \alpha\langle V_S(\alpha)\xi|\eta\rangle, \quad \forall \xi, \eta \in \mathcal{D}; \alpha \geqslant 0.$$

Apply the Cauchy-Schwarz inequality $(\xi = \eta) \Rightarrow \forall z \in \mathbb{C}, \alpha \geqslant 0$,

$$\alpha|\langle V_S(\alpha)\xi|\xi\rangle| \leqslant 2\max\{\|(T - z)\xi\|, \|(T^\dagger - \bar{z})\xi\|\} \max\{\|V_S(\alpha)\xi\|, \|V_S(\alpha)^*\xi\|\}. \quad (1)$$

Consequences:

- if $T = T^\dagger \Rightarrow \sigma_p(T) = \emptyset$.
- If V_S uniformly bounded, i.e. $\|V_S(\alpha)\| \leqslant M$, $\forall\alpha \geqslant 0$, (e.g. V_S isometries or contractions), by (1) $\lim_{\alpha\to\infty} |\langle V_S(\alpha)\xi|\xi\rangle| = 0, \quad \forall\xi \in \mathcal{H}$.
- V_S semigroup of contractions (i.e., $\|V_S(\alpha)\| \leqslant 1$), $\forall\, \alpha \geqslant 0 \Rightarrow$ every eigenvalue of S has negative real part.

Some consequences of (CR.3)

Assume $[S, T] = \mathbb{1}_{\mathcal{D}}$ in quasi-strong sense; i.e.

$$\langle V_S(\alpha) T\xi|\eta\rangle - \langle V_S(\alpha)\xi|T^\dagger\eta\rangle = \alpha\langle V_S(\alpha)\xi|\eta\rangle, \quad \forall \xi, \eta \in \mathcal{D}; \alpha \geqslant 0.$$

Apply the Cauchy-Schwarz inequality $(\xi = \eta) \Rightarrow \forall z \in \mathbb{C}, \alpha \geqslant 0,$

$$\alpha|\langle V_S(\alpha)\xi|\xi\rangle| \leqslant 2\max\{\|(T-z)\xi\|, \|(T^\dagger - \bar{z})\xi\|\} \max\{\|V_S(\alpha)\xi\|, \|V_S(\alpha)^*\xi\|\}. \quad (1)$$

Consequences:

- if $T = T^\dagger \Rightarrow \sigma_p(T) = \emptyset$.
- If V_S uniformly bounded, i.e. $\|V_S(\alpha)\| \leqslant M$, $\forall \alpha \geqslant 0$, (e.g. V_S isometries or contractions), by (1) $\lim_{\alpha\to\infty} |\langle V_S(\alpha)\xi|\xi\rangle| = 0, \quad \forall \xi \in \mathcal{H}$.
- V_S semigroup of contractions (i.e., $\|V_S(\alpha)\| \leqslant 1$), $\forall\, \alpha \geqslant 0 \Rightarrow$ every eigenvalue of S has negative real part.

Corollary (Miyamoto's result) If the generator X of V_S has the form $X = iH$ where H is a self-adjoint operator, then $\sigma_p(H) = \emptyset$.

Time operators

Schmüdgen studied pairs of operators (T, H)
T symmetric, H self-adjoint s.t.

- $e^{-itH}D(T) \subseteq D(T)$;
- $Te^{-itH}\xi = e^{-itH}(T+t)\xi$

T, H regarded as members of $\mathcal{L}^{\dagger}(\mathcal{D}, \mathcal{H})$ with $\mathcal{D} = D(T) \cap D(S)$.
This is equivalent to the operators $T, S := iH$ satisfy $[S, T] = \mathbb{1}$ in quasi-strong sense

Definition

T is time operator for H if (T, H) satisfies the two conditions above.

Schmüdgen studied pairs of operators (T, H)
T symmetric, H self-adjoint s.t.

- $e^{-itH}D(T) \subseteq D(T)$;
- $Te^{-itH}\xi = e^{-itH}(T + t)\xi$

T, H regarded as members of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ with $\mathcal{D} = D(T) \cap D(S)$.
This is equivalent to the operators $T, S := iH$ satisfy $[S, T] = \mathbb{1}$ in quasi-strong sense

Definition

T is time operator for H if (T, H) satisfies the two conditions above.

In many examples, mostly taken from Physics, H is a semibounded operator (H the Hamiltionian of some physical system).

Theorem

(Arai) $H = H^*$, H semibounded. Then no time operator T of H can be essentially self-adjoint.

Indeed, the spectrum $\sigma(T)$ is one of the following sets

Theorem

(Arai) $H = H^*$, H semibounded. Then no time operator T of H can be essentially self-adjoint.

Indeed, the spectrum $\sigma(T)$ is one of the following sets

- $\mathbb{C}$, if H is bounded

Theorem

(Arai) $H = H^*$, H semibounded. Then no time operator T of H can be essentially self-adjoint.

Indeed, the spectrum $\sigma(T)$ is one of the following sets

- $\mathbb{C}$, if H is bounded
- $\mathbb{C}$ or $\overline{\Pi_+} = \{z \in \mathbb{C} : \Im z \geqslant 0\}$ if H is bounded below

Theorem

(Arai) $H = H^*$, H semibounded. Then no time operator T of H can be essentially self-adjoint.

Indeed, the spectrum $\sigma(T)$ is one of the following sets

- $\mathbb{C}$, if H is bounded
- $\mathbb{C}$ or $\overline{\Pi_+} = \{z \in \mathbb{C} : \Im z \geqslant 0\}$ if H is bounded below
- $\mathbb{C}$ or $\overline{\Pi_-} = \{z \in \mathbb{C} : \Im z \leqslant 0\}$ if H is bounded above.

Theorem

(Arai) $H = H^*$, H semibounded. Then no time operator T of H can be essentially self-adjoint.

Indeed, the spectrum $\sigma(T)$ is one of the following sets

- $\mathbb{C}$, if H is bounded
- $\mathbb{C}$ or $\overline{\Pi_+} = \{z \in \mathbb{C} : \Im z \geqslant 0\}$ if H is bounded below
- $\mathbb{C}$ or $\overline{\Pi_-} = \{z \in \mathbb{C} : \Im z \leqslant 0\}$ if H is bounded above.

Question: Which one is realized, depending on properties of H?

Examples

I interval of the real line. Denote by q the multiplication operator on $L^2(I)$ by the variable $x \in I$
q is selfadjoint.

Examples

I interval of the real line. Denote by q the multiplication operator on $L^2(I)$ by the variable $x \in I$
q is selfadjoint.

Let p be the operator on $L^2(I)$ defined as follows:
$D(p) := C_c^\infty(I)$
$(pg)(x) := -ig'(x), \quad g \in D(p)$

Examples

I interval of the real line. Denote by q the multiplication operator on $L^2(I)$ by the variable $x \in I$
q is selfadjoint.

Let p be the operator on $L^2(I)$ defined as follows:
$D(p) := C_c^\infty(I)$
$(pg)(x) := -ig'(x), \quad g \in D(p)$

CASE 1: $I = [0, \infty)$ q is positive; $-p$ is time operator of q and $\sigma(-p) = \overline{\Pi_+}$.

Examples

I interval of the real line. Denote by q the multiplication operator on $L^2(I)$ by the variable $x \in I$
q is selfadjoint.

Let p be the operator on $L^2(I)$ defined as follows:
$D(p) := C_c^\infty(I)$
$(pg)(x) := -ig'(x), \quad g \in D(p)$

CASE 1: $I = [0, \infty)$ q is positive; $-p$ is time operator of q and $\sigma(-p) = \overline{\Pi}_+$.

CASE 2: $I = (-L/2, L/2),\ L > 0$
q is a bounded self-adjoint operator.
$-p$ is a time operator of q and $\sigma(-p) = \mathbb{C}$.

Relaxing assumptions

The assumption $e^{-itH}D(T) \subseteq D(T)$ is quite strong.
Try to relax it! (Bagarello, Inoue, CT, 2012)

Definition

$\{S, T\} \subset \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$; $S = S^\dagger$; $T = T^\dagger$
$\{S,T\}$ satisfy weak Weyl commutation relation if $\exists$ H self-adjoint extension of S such that

The assumption $e^{-itH}D(T) \subseteq D(T)$ is quite strong.
Try to relax it! (Bagarello, Inoue, CT, 2012)

Definition

$\{S, T\} \subset \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$; $S = S^\dagger$; $T = T^\dagger$
$\{S,T\}$ satisfy weak Weyl commutation relation if $\exists$ H self-adjoint extension of S such that

- $D(\overline{T}) \subset D(H)$

Relaxing assumptions

The assumption $e^{-itH}D(T) \subseteq D(T)$ is quite strong.
Try to relax it! (Bagarello, Inoue, CT, 2012)

Definition

$\{S, T\} \subset \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$; $S = S^\dagger$; $T = T^\dagger$
$\{S,T\}$ satisfy weak Weyl commutation relation if $\exists$ H self-adjoint extension of S such that

- $D(\overline{T}) \subset D(H)$
- $\langle e^{-itH}\xi | T\eta\rangle = \langle (T + t)\xi | e^{itH}\eta\rangle, \quad \forall \xi, \eta \in \mathcal{D}, \forall t \in \mathbb{R}.$

$H :=$ weak Weyl extension of S.

The assumption $e^{-itH}D(T) \subseteq D(T)$ is quite strong.
Try to relax it! (Bagarello, Inoue, CT, 2012)

Definition

$\{S, T\} \subset \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$; $S = S^\dagger$; $T = T^\dagger$
$\{S,T\}$ satisfy weak Weyl commutation relation if $\exists$ H self-adjoint extension of S such that

- $D(\overline{T}) \subset D(H)$
- $\langle e^{-itH}\xi | T\eta \rangle = \langle (T + t)\xi | e^{itH}\eta \rangle, \quad \forall \xi, \eta \in \mathcal{D}, \forall t \in \mathbb{R}.$

$H :=$ weak Weyl extension of S. Results:

Relaxing assumptions

The assumption $e^{-itH}D(T) \subseteq D(T)$ is quite strong.
Try to relax it! (Bagarello, Inoue, CT, 2012)

Definition

$\{S, T\} \subset \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$; $S = S^\dagger$; $T = T^\dagger$
$\{S,T\}$ satisfy weak Weyl commutation relation if $\exists$ H self-adjoint extension of S such that

- $D(\overline{T}) \subset D(H)$
- $\langle e^{-itH}\xi | T\eta\rangle = \langle (T + t)\xi | e^{itH}\eta\rangle, \quad \forall \xi, \eta \in \mathcal{D}, \forall t \in \mathbb{R}.$

$H :=$ weak Weyl extension of S. Results:

- Suppose T essentially self-adj. Then $\{H, \overline{T}\}$ satisfy the Weyl commutation relations

$$e^{itH}e^{-is\overline{T}} = e^{ist}e^{-is\overline{T}}e^{itH}, \quad \forall s, t \in \mathbb{R}$$

Relaxing assumptions

The assumption $e^{-itH}D(T) \subseteq D(T)$ is quite strong.
Try to relax it! (Bagarello, Inoue, CT, 2012)

Definition

$\{S, T\} \subset \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$; $S = S^\dagger$; $T = T^\dagger$
$\{S,T\}$ satisfy weak Weyl commutation relation if $\exists$ H self-adjoint extension of S such that

- $D(\overline{T}) \subset D(H)$
- $\langle e^{-itH}\xi | T\eta\rangle = \langle (T+t)\xi | e^{itH}\eta\rangle, \quad \forall \xi, \eta \in \mathcal{D}, \forall t \in \mathbb{R}.$

$H :=$ weak Weyl extension of S. Results:

- Suppose T essentially self-adj. Then $\{H, \overline{T}\}$ satisfy the Weyl commutation relations

$$e^{itH}e^{-is\overline{T}} = e^{ist}e^{-is\overline{T}}e^{itH}, \quad \forall s, t \in \mathbb{R}$$

- If H is semibounded then T is not essentially selfadj.

Relaxing assumptions

The assumption $e^{-itH}D(T) \subseteq D(T)$ is quite strong.
Try to relax it! (Bagarello, Inoue, CT, 2012)

Definition

$\{S, T\} \subset \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$; $S = S^\dagger$; $T = T^\dagger$
$\{S,T\}$ satisfy weak Weyl commutation relation if $\exists$ H self-adjoint extension of S such that

- $D(\overline{T}) \subset D(H)$
- $\langle e^{-itH}\xi | T\eta\rangle = \langle (T+t)\xi | e^{itH}\eta\rangle, \quad \forall \xi, \eta \in \mathcal{D}, \forall t \in \mathbb{R}.$

$H :=$ weak Weyl extension of S. Results:

- Suppose T essentially self-adj. Then $\{H, \overline{T}\}$ satisfy the Weyl commutation relations

$$e^{itH}e^{-is\overline{T}} = e^{ist}e^{-is\overline{T}}e^{itH}, \quad \forall s, t \in \mathbb{R}$$

- If H is semibounded then T is not essentially selfadj.
- H semibounded, $\mathcal{D}^\infty(T^*) \subset D(\overline{T})$. Then $\sigma(T) = \mathbb{C}$.

Nonlinear extension

Generalization of condition (CR2)
$\{\varphi_n\}$, $\{\psi_n\}$ two biorthogonal bases contained in $\mathcal{D}$ and

$$X = \sum_{k=0}^{\infty} \alpha_k (\psi_k \otimes \overline{\varphi_k}),$$

$\{\alpha_n\}$ a sequence of positive real numbers.
Assume $\mathcal{D} \subset D(X)$.

Nonlinear extension

Generalization of condition (CR2)
$\{\varphi_n\}$, $\{\psi_n\}$ two biorthogonal bases contained in $\mathcal{D}$ and

$$X = \sum_{k=0}^{\infty} \alpha_k (\psi_k \otimes \overline{\varphi_k}),$$

$\{\alpha_n\}$ a sequence of positive real numbers.
Assume $\mathcal{D} \subset D(X)$.

S and T satisfy the nonlinear CR.2 if, $\forall \xi, \eta \in \mathcal{D}$,

$$\langle T\,\xi | S^\dagger\,\eta\rangle - \langle S\,\xi | T^\dagger\,\eta\rangle = \langle \xi | X\,\eta\rangle, \tag{2}$$

S and $T^\dagger$ act as raising operators on bases vectors; $S^\dagger$ and T as lowering operators, but squares of eigenvalues do not depend linearly on n
An analysis similar to the case $X = 1\!\!1_{\mathcal{D}}$ can be done; some results extend (with more constraints) to this *nonlinear* situation.