

# On the analytical solvability of (polynomial) 1D Schrödinger equations

**André Voros**  
CEA, Institut de Physique Théorique de Saclay

August 29, 2012

J. Phys. A (2012, special issue to honor Prof. S. Dowker, in press) [arXiv:1202.3100 \[math-ph\]](https://arxiv.org/abs/1202.3100)  
RIMS Kôkyûroku **1424** (2005) 214–231 [\[math-ph/0412041\]](https://arxiv.org/abs/math-ph/0412041)

$$\left( -\frac{d^2}{dq^2} + [V(q) + \lambda] \right) \psi(q) = 0$$

(2nd order Ordinary Differential Equation, of Sturm–Liouville type)

$$\left( -\frac{d^2}{dq^2} + [V(q) + \lambda] \right) \psi(q) = 0$$

$\uparrow$                      $\uparrow$   
 $\{+ q^N + v_1 q^{N-1} + \dots + v_{N-1} q\}$        $\{-E\}$

Notations :  $\vec{v} = (v_1, \dots, v_{N-1})$  Degree =  $N$

- Recent view (in 1 dimension):

### AN EXACTLY SOLVABLE PROBLEM IN ANY DEGREE

by an **exact WKB method** (cf. Balian–Bloch, Zinn-Justin, Sibuya):

- semiclassical analysis using **zeta-regularization**
- exact** Bohr–Sommerfeld quantization conditions of the form

$$\Sigma(E) = k + \frac{1}{2} \implies E_k,$$

selfconsistent ( $\approx$  Bethe Ansatz).

$$\left( -\frac{d^2}{dq^2} + [V(q) + \lambda] \right) \psi(q) = 0$$

$\uparrow$        $\uparrow$   
 $\{+ q^N + v_1 q^{N-1} + \dots + v_{N-1} q\}$        $\{-E\}$

Notations :  $\vec{v} = (v_1, \dots, v_{N-1})$  Degree =  $N$

• Conjugate equations:

$$V^{[\ell]}(q) \stackrel{\text{def}}{=} e^{-i\ell\varphi} V(e^{-i\ell\varphi/2} q), \quad \lambda^{[\ell]} \stackrel{\text{def}}{=} e^{-i\ell\varphi} \lambda$$

$$\text{for } \ell = 0, 1, \dots, L-1 \pmod{L} \quad \text{with} \quad \boxed{\varphi \stackrel{\text{def}}{=} \frac{4\pi}{N+2}}$$

$$\text{Number of distinct conjugates : } L = \begin{cases} N+2 & \text{generically} \\ \frac{N}{2} + 1 & \text{for even polynomials } V(q) \end{cases}$$

- $E \rightarrow +\infty$  expansions:

Classical action:  $\oint_{\{p^2+V(q)=E\}} \frac{p dq}{2\pi} \sim b_\mu E^\mu, \quad \boxed{\mu \stackrel{\text{def}}{=} \frac{1}{2} + \frac{1}{N}}$  (growth order)

Semiclassical quantization condition (*Bohr–Sommerfeld expansion*):

$$\sum_\alpha b_\alpha E_k^\alpha \sim k + \frac{1}{2} \quad \text{for integer } k \rightarrow +\infty \quad \left( \alpha = \mu, \mu - \frac{1}{N}, \mu - \frac{2}{N}, \dots \right)$$

$\Downarrow$

- **(Generalized) spectral zeta functions**

$$Z^\pm(s, \lambda) \stackrel{\text{def}}{=} \sum_{k \text{ even/odd}} (E_k + \lambda)^{-s} \quad (\text{convergent for } \operatorname{Re} s > \mu) \quad \begin{array}{l} + : \text{Neumann} \\ - : \text{Dirichlet} \end{array}$$

and  $Z \equiv Z^+ + Z^-$  ((full) : all **meromorphic** in  $s$ -plane, **regular at  $s=0$** )

- **Spectral determinants** (zeta-regularized)

$$D^\pm(\lambda) \equiv D(\lambda | \mathcal{E}_\pm) \stackrel{\text{def}}{=} \exp[-\partial_s Z^\pm(s, \lambda)]_{s=0} \quad (\begin{array}{l} \text{Neumann} \\ \text{Dirichlet} \end{array} \text{ determinant})$$

and  $D \equiv D^+ D^-$  (full determinant) : all **entire** functions, of order  $\mu$

Two key properties of the *zeta-regularized*  $\log D$  (and likewise,  $\log D^\pm$ ):

- a **structure formula**:

$$\log D(\lambda) \equiv \lim_{K \rightarrow +\infty} \left\{ \sum_{k < K} \log(E_k + \lambda) + \frac{1}{2} \log(E_K + \lambda) - \sum_{\{\alpha > 0\}} b_\alpha E_K^\alpha \left[ \log E_K - \frac{1}{\alpha} \right] \right\},$$

*counterterms*

- a **canonical** large- $\lambda$  (*generalized Stirling*) expansion

$$\log D(\lambda) \sim \sum_{\alpha} a_{\alpha} \{\lambda^{\alpha}\} \quad \left( \alpha = \mu, \mu - \frac{1}{N}, \mu - \frac{2}{N}, \dots \right) :$$

$$\{\lambda^{\alpha}\} \stackrel{\text{def}}{=} \lambda^{\alpha} \quad (\alpha \notin \mathbb{N}), \quad \{\lambda^1\} \stackrel{\text{def}}{=} \lambda(\log \lambda - 1), \quad \{\lambda^0\} \stackrel{\text{def}}{=} \log \lambda;$$

*banned:* pure  $\lambda^n$  ( $n \in \mathbb{N}$ ) terms, including **additive constants** ( $\propto \lambda^0$ ).

Exact solution  $\varphi_\lambda(q)$ , recursive for  $q \rightarrow \infty$  ([Cartoonish WKB approximation](#)).

$$\psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} \exp \underbrace{\int_q^{+\infty} \Pi_\lambda(\tilde{q}) d\tilde{q}}_{improper \text{ action integral}}, \quad \Pi_\lambda(q) \stackrel{\text{def}}{=} (V(q) + \lambda)^{1/2} \quad (\text{classical momentum})$$
$$(\Pi_\lambda(\tilde{q}) \sim \tilde{q}^{N/2}).$$

$$\psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} \exp \underbrace{\int_q^{+\infty} \Pi_\lambda(\tilde{q}) d\tilde{q}}_{\text{improper action integral}} , \quad \Pi_\lambda(q) \stackrel{\text{def}}{=} (V(q) + \lambda)^{1/2} \quad (\text{classical momentum})$$

$$(\Pi_\lambda(\tilde{q}) \sim \tilde{q}^{N/2}).$$

$$\text{Naive idea: } = \left[ \underbrace{\int_q^{+\infty} (V(\tilde{q}) + \lambda)^{1/2-s} d\tilde{q}}_{I_q(s, \lambda)} \right]_{s \rightsquigarrow 0} \quad (\text{convergent for } \operatorname{Re} s > \mu)$$

$$(\text{analytical continuation in } s)$$

OK when  $I_q(s, \lambda)$  is **regular** at  $s = 0$ ; but in general,

$$(V(q) + \lambda)^{1/2-s} \sim \sum_{\rho} \beta_{\rho}(s; \vec{v}) q^{\rho - Ns} \quad (\rho = \frac{N}{2}, \frac{N}{2}-1, \dots) \quad (q \rightarrow +\infty)$$

$$\Rightarrow I_q(s, \lambda) \sim - \sum_{\rho} \beta_{\rho}(s; \vec{v}) \frac{q^{\rho+1-Ns}}{\rho + 1 - Ns} \quad (\text{singular expansion})$$

$$\Rightarrow I_q(s, \lambda) \text{ has at most a simple pole at } s = 0, \text{ of residue } \boxed{\frac{1}{N} \beta_{-1}(s=0; \vec{v})};$$

$$\beta_{-1}(\vec{v}) = \sum_{\{r_j \geq 0\}} \delta_{\sum_{j=1}^{N-1} j r_j, 1+N/2} \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2} - \sum_{j=1}^{N-1} r_j)} \frac{v_1^{r_1} \cdots v_{N-1}^{r_{N-1}}}{r_1! \cdots r_{N-1}!} \quad (\text{for } N \neq 2).$$

QUANTUM       $\longleftrightarrow$       CLASSICAL

zeta function

$$\begin{aligned} Z(s, \lambda) &= \text{Tr} \left( -\frac{d^2}{dq^2} + V(q) + \lambda \right)^{-s} \\ &= \sum_k (E_k + \lambda)^{-s} \end{aligned}$$

determinant

$$\begin{aligned} D(\lambda) &= \text{"formally"} \prod_k (\lambda + E_k) \\ D(\lambda) &\stackrel{\text{def}}{=} \exp \{-\partial_s Z(s, \lambda)|_{s=0}\} \end{aligned}$$

QUANTUM  $\longleftrightarrow$  CLASSICAL

zeta function

zeta function?

$$Z(s, \lambda) = \text{Tr} \left( -\frac{d^2}{dq^2} + V(q) + \lambda \right)^{-s}$$

$$= \sum_k (E_k + \lambda)^{-s}$$

determinant

$$D(\lambda) = \underset{k}{\prod} (\lambda + E_k) \quad \text{formally}$$

$$D(\lambda) \stackrel{\text{def}}{=} \exp \{-\partial_s Z(s, \lambda)|_{s=0}\}$$

QUANTUM  $\longleftrightarrow$  CLASSICAL

zeta function

zeta function

$$Z(s, \lambda) = \text{Tr} \left( -\frac{d^2}{dq^2} + V(q) + \lambda \right)^{-s} \quad Z_{\text{cl}}(s, \lambda) \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \frac{dq dp}{2\pi} \left( p^2 + V(q) + \lambda \right)^{-s}$$

$$= \sum_k (E_k + \lambda)^{-s}$$

determinant

$$D(\lambda) = \underset{k}{\prod} (\lambda + E_k)$$

$$D(\lambda) \stackrel{\text{def}}{=} \exp \{-\partial_s Z(s, \lambda)|_{s=0}\}$$

QUANTUM       $\longleftrightarrow$       CLASSICAL

zeta function

zeta function

$$\begin{aligned} Z(s, \lambda) &= \text{Tr} \left( -\frac{d^2}{dq^2} + V(q) + \lambda \right)^{-s} \\ &= \sum_k (E_k + \lambda)^{-s} \end{aligned} \quad \begin{aligned} Z_{\text{cl}}(s, \lambda) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \frac{dq dp}{2\pi} \left( p^2 + V(q) + \lambda \right)^{-s} \\ &= \frac{\Gamma(s-1/2)}{2\sqrt{\pi} \Gamma(s)} \int_{-\infty}^{+\infty} (V(q) + \lambda)^{1/2-s} dq \end{aligned}$$

determinant

determinant?

$$D(\lambda) = \underset{k}{\overset{\text{formally}}{\prod}} (\lambda + E_k)$$

$$D(\lambda) \stackrel{\text{def}}{=} \exp \{-\partial_s Z(s, \lambda)|_{s=0}\}$$

$$D_{\text{cl}}(\lambda) \stackrel{\text{def}}{=} \exp \{-\partial_s Z_{\text{cl}}(s, \lambda)|_{s=0}\}$$

QUANTUM  $\longleftrightarrow$  CLASSICAL

zeta function

$$\begin{aligned} Z(s, \lambda) &= \text{Tr} \left( -\frac{d^2}{dq^2} + V(q) + \lambda \right)^{-s} \\ &= \sum_k (E_k + \lambda)^{-s} \end{aligned} \quad \begin{aligned} Z_{\text{cl}}(s, \lambda) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \frac{dq dp}{2\pi} \left( p^2 + V(q) + \lambda \right)^{-s} \\ &= \frac{\Gamma(s-1/2)}{2\sqrt{\pi} \Gamma(s)} \int_{-\infty}^{+\infty} (V(q) + \lambda)^{1/2-s} dq \end{aligned}$$

determinant

$$\begin{aligned} D(\lambda) &= \text{"formally"} \prod_k (\lambda + E_k) \\ D(\lambda) &\stackrel{\text{def}}{=} \exp \{-\partial_s Z(s, \lambda)|_{s=0}\} \end{aligned} \quad \begin{aligned} D_{\text{cl}}(\lambda) &= \text{"formally"} \exp \left\{ \int_{-\infty}^{+\infty} (V(q) + \lambda)^{1/2} dq \right\} \\ D_{\text{cl}}(\lambda) &\stackrel{\text{def}}{=} \exp \{-\partial_s Z_{\text{cl}}(s, \lambda)|_{s=0}\} \end{aligned}$$

$$\log D_{\text{cl}}(\lambda) = \int_{-\infty}^{+\infty} (V(q) + \lambda)^{1/2} dq = 2 \int_0^{+\infty} (V(q) + \lambda)^{1/2} dq \quad ?$$

$$\frac{d}{d\lambda} \log D_{\text{cl}}(\lambda) = \int_0^{+\infty} (V(q) + \lambda)^{-1/2} dq$$

&

$$\begin{aligned} \log D_{\text{cl}}(\lambda) &\sim \text{CANONICAL} \quad \text{for } \lambda \rightarrow +\infty \\ (\iff & \text{devoid of pure } \lambda^0 \text{ terms in large-}\lambda \text{ expansion}) \end{aligned}$$

specify **improper** action integral completely:

$$\begin{aligned} \int_0^{+\infty} \Pi_\lambda(q) dq &\stackrel{\text{def}}{=} \text{FP}_{s=0} I_0(s, \lambda) + \frac{2}{N}(1 - \log 2) \beta_{-1}(\vec{v}), \\ \Pi_\lambda(q) &= (V(q) + \lambda)^{1/2}, \quad I_0(s, \lambda) = \int_0^{+\infty} (V(q) + \lambda)^{1/2-s} dq. \end{aligned}$$

Simplest examples.

$$\int_0^{+\infty} (q^N + \lambda)^{1/2} dq = -(2\sqrt{\pi})^{-1} \Gamma(1 + \frac{1}{N}) \Gamma(-\frac{1}{2} - \frac{1}{N}) \lambda^{\mu} \quad (N \neq 2) \quad \boxed{\mu \stackrel{\text{def}}{=} \frac{1}{2} + \frac{1}{N}}$$

$$\int_0^{+\infty} (q^2 + \lambda)^{1/2} dq = -\frac{1}{4} \lambda (\log \lambda - 1) \quad (N = 2 : \beta_{-1}(\vec{v}) \neq 0)$$

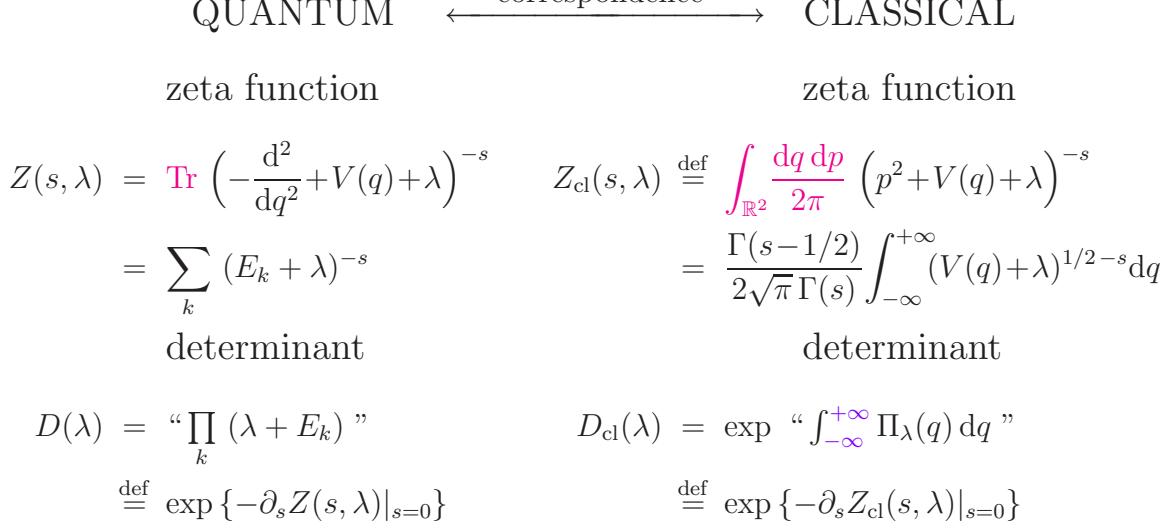
Even quartic oscillator ( $v, \lambda \geq 0$  for simplicity):

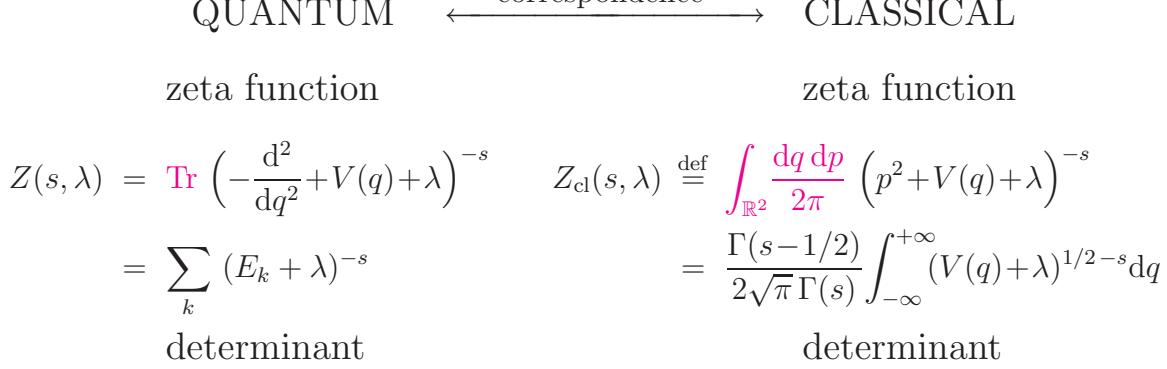
$$\int_0^{+\infty} (q^4 + vq^2)^{1/2} dq = -\frac{1}{3} v^{3/2}$$

$$\int_0^{+\infty} (q^4 + vq^2 + \lambda)^{1/2} dq =$$

$$(v \geq 2\sqrt{\lambda}) : \quad = \frac{1}{3}(v + 2\sqrt{\lambda})^{1/2} [2\sqrt{\lambda} K(k) - vE(k)], \quad k = \left( \frac{v - 2\sqrt{\lambda}}{v + 2\sqrt{\lambda}} \right)^{1/2};$$

$$(v \leq 2\sqrt{\lambda}) : \quad = \frac{1}{3} \lambda^{1/4} [(2\sqrt{\lambda} + v)K(\tilde{k}) - 2vE(\tilde{k})], \quad \tilde{k} = \frac{(2\sqrt{\lambda} - v)^{1/2}}{2 \lambda^{1/4}}.$$





$$D(\lambda) = \left( \prod_k (\lambda + E_k) \right)^{-1} \stackrel{\text{def}}{=} \exp \left\{ -\partial_s Z(s, \lambda) \Big|_{s=0} \right\}$$

$$D_{\text{cl}}(\lambda) = \exp \left[ \int_{-\infty}^{+\infty} \Pi_\lambda(q) dq \right]$$

$$\stackrel{\text{def}}{=} \exp \left\{ -\partial_s Z_{\text{cl}}(s, \lambda) \Big|_{s=0} \right\}$$

determinant

determinant

WKB identities

$$\begin{aligned} D_{\text{cl}}^-(\lambda) &\equiv \Pi_\lambda(0)^{-1/2} \exp \int_0^{+\infty} \Pi_\lambda(q) dq \\ &\equiv [\psi_\lambda]_{\text{WKB}}(0) \end{aligned}$$

$$\begin{aligned} D_{\text{cl}}^+(\lambda) &\equiv \Pi_\lambda(0)^{+1/2} \exp \int_0^{+\infty} \Pi_\lambda(q) dq \\ &\equiv -[\psi'_\lambda]_{\text{WKB}}(0) \end{aligned}$$

zeta function

zeta function

$$\begin{aligned}
 Z(s, \lambda) &= \text{Tr} \left( -\frac{d^2}{dq^2} + V(q) + \lambda \right)^{-s} \\
 &= \sum_k (E_k + \lambda)^{-s} \\
 &\quad \text{determinant}
 \end{aligned}
 \qquad
 \begin{aligned}
 Z_{\text{cl}}(s, \lambda) &\stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \frac{dq dp}{2\pi} \left( p^2 + V(q) + \lambda \right)^{-s} \\
 &= \frac{\Gamma(s-1/2)}{2\sqrt{\pi}\Gamma(s)} \int_{-\infty}^{+\infty} (V(q) + \lambda)^{1/2-s} dq \\
 &\quad \text{determinant}
 \end{aligned}$$

$$\begin{aligned}
 D(\lambda) &= \text{“} \prod_k (\lambda + E_k) \text{”} \\
 &\stackrel{\text{def}}{=} \exp \{ -\partial_s Z(s, \lambda) |_{s=0} \}
 \end{aligned}$$

$$\begin{aligned}
 D_{\text{cl}}(\lambda) &= \exp \text{“} \int_{-\infty}^{+\infty} \Pi_\lambda(q) dq \text{”} \\
 &\stackrel{\text{def}}{=} \exp \{ -\partial_s Z_{\text{cl}}(s, \lambda) |_{s=0} \}
 \end{aligned}$$

**basic exact identities****WKB identities**

$$D^-(\lambda) \equiv \psi_\lambda(0)$$

$$\begin{aligned}
 D_{\text{cl}}^-(\lambda) &\equiv \Pi_\lambda(0)^{-1/2} \exp \int_0^{+\infty} \Pi_\lambda(q) dq \\
 &\equiv [\psi_\lambda]_{\text{WKB}}(0)
 \end{aligned}$$

$$D^+(\lambda) \equiv -\psi'_\lambda(0)$$

$$\begin{aligned}
 D_{\text{cl}}^+(\lambda) &\equiv \Pi_\lambda(0)^{+1/2} \exp \int_0^{+\infty} \Pi_\lambda(q) dq \\
 &\equiv -[\psi'_\lambda]_{\text{WKB}}(0)
 \end{aligned}$$

$$\psi_\lambda(0) \equiv D^-(\lambda), \quad \psi'_\lambda(0) \equiv -D^+(\lambda) \quad (*)$$

for the **Dirichlet**, resp. **Neumann**, spectral determinants  $D^\pm(\lambda)$  on  $[0, +\infty)$ .

Of course, full (translation-covariant) theorem is

$$\psi_\lambda(q) \equiv D_q^-(\lambda), \quad \psi'_\lambda(q) \equiv -D_q^+(\lambda)$$

for the **Dirichlet**, resp. **Neumann**, spectral determinants  $D_q^\pm(\lambda)$  on  $[q, +\infty)$ .

All subject to:  $\psi_\lambda(q) =$  the solution recessive for  $q \rightarrow +\infty$ , **canonically** normalized.

$$\psi_\lambda(0) \equiv D^-(\lambda), \quad \psi'_\lambda(0) \equiv -D^+(\lambda) \quad (*)$$

1) Apply  $\cdot \stackrel{\text{def}}{=} \frac{d}{d\lambda}$  to  $\log(*)$  to get (with  $' \stackrel{\text{def}}{=} \frac{d}{dq}$ )

$$\dot{\psi}_\lambda(0)/\psi_\lambda(0) \equiv Z^-(1, \lambda), \quad \dot{\psi}'_\lambda(0)/\psi'_\lambda(0) \equiv Z^+(1, \lambda), \quad (**)$$

using  $\log D^\pm(\lambda) \stackrel{\text{def}}{=} -\partial_s Z^\pm(s, \lambda)_{s=0}$  then  $[\partial_s Z^\pm(s, \lambda)_{s=0}]' \equiv -Z^\pm(1, \lambda)$ .

2) Apply  $\int d\lambda$  to  $(**)$  to obtain  $\log(*)$ , by **fully controlling all integration constants**.

( $N = 1, 2$  need **two** differentiations, resp. integrations, in  $\lambda$ ;  $N = 2$  case reestablishes the Stirling formula for  $1/\Gamma(\lambda)$ .)

$$\psi_\lambda(0) \equiv D^-(\lambda), \quad \psi'_\lambda(0) \equiv -D^+(\lambda) \quad (*)$$

1) Apply  $\cdot \stackrel{\text{def}}{=} \frac{d}{d\lambda}$  to  $\log(*)$  to get (with  $' \stackrel{\text{def}}{=} \frac{d}{dq}$ )

$$\dot{\psi}_\lambda(0)/\psi_\lambda(0) \equiv Z^-(1, \lambda), \quad \dot{\psi}'_\lambda(0)/\psi'_\lambda(0) \equiv Z^+(1, \lambda). \quad (**)$$

Proof of (\*\*): a **trace formula** (pair), for the resolvent traces  $\text{Tr}(-\frac{d^2}{dq^2} + V(q) + \lambda)^{-1}_{\pm}$  on  $[0, +\infty)$  with  $\begin{cases} \text{Neumann} \\ \text{Dirichlet} \end{cases}$  boundary conditions.

- spectral evaluation:  $\text{Tr}_{\pm} = \sum_{k \begin{cases} \text{even} \\ \text{odd} \end{cases}} (E_k + \lambda)^{-1} \equiv Z^{\pm}(1, \lambda); \quad (\text{RHS})$
- Green's function evaluation, using the known (in 1D) integral kernels of the resolvents  $G_\lambda^{\pm}(q, \tilde{q}) = W(\psi_\lambda, \psi_\lambda^{\pm})^{-1} \psi_\lambda^{\pm}(\min(q, \tilde{q})) \psi_\lambda(\max(q, \tilde{q})) \quad (q, \tilde{q} \in [0, +\infty), \psi_\lambda^{\pm}(q) \stackrel{\text{def}}{=} \begin{cases} \text{Neumann} \\ \text{Dirichlet} \end{cases} \text{ solution}):$

$$\text{Tr}_{\pm} = \int_0^\infty G_\lambda^{\pm}(q, q) dq = W(\psi_\lambda, \psi_\lambda^{\pm})^{-1} \int_0^\infty \psi_\lambda^{\pm}(q) \psi_\lambda(q) dq;$$

now the subtraction of  $\dot{\psi}_\lambda[-\psi_\lambda'' + (V(q) + \lambda)\psi_\lambda] = 0$  from  $\psi_\lambda^{\pm}[-\psi_\lambda'' + (V(q) + \lambda)\psi_\lambda]' = 0$  yields  $\psi_\lambda^{\pm}\psi_\lambda = [W(\psi_\lambda^{\pm}, \dot{\psi}_\lambda)]'$  hence

$$\text{Tr}_{\pm} = W(\psi_\lambda, \psi_\lambda^{\pm})^{-1} \left[ W(\psi_\lambda^{\pm}, \dot{\psi}_\lambda) \right]_0^\infty = -W(\psi_\lambda, \psi_\lambda^{\pm})^{-1} W(\psi_\lambda^{\pm}, \dot{\psi}_\lambda)_{q=0}. \quad (\text{LHS})$$

$$\psi_\lambda(0) \equiv D^-(\lambda), \quad \psi'_\lambda(0) \equiv -D^+(\lambda) \quad (*)$$

1) Apply  $\cdot \stackrel{\text{def}}{=} \frac{d}{d\lambda}$  to  $\log(*)$  to get

$$\dot{\psi}_\lambda(0)/\psi_\lambda(0) \equiv Z^-(1, \lambda), \quad \dot{\psi}'_\lambda(0)/\psi'_\lambda(0) \equiv Z^+(1, \lambda), \quad (**)$$

using  $\log D^\pm(\lambda) \stackrel{\text{def}}{=} -\partial_s Z^\pm(s, \lambda)_{s=0}$  then  $[\partial_s Z^\pm(s, \lambda)_{s=0}] \cdot \equiv -Z^\pm(1, \lambda)$ .

2) Apply  $\int d\lambda$  to  $(**)$  to obtain  $\log(*)$ , by fully controlling all integration constants:

$$\log \psi_\lambda(0) \equiv \log D^-(\lambda) + \text{const.} , \quad \log[-\psi'_\lambda(0)] \equiv \log D^+(\lambda) + \text{const.} ,$$

but  $\log \psi_\lambda$  and  $\log D^\pm(\lambda)$  have canonical ( $\lambda \rightarrow +\infty$ ) expansions: only const. ( $\propto \lambda^0$ ) shift here  $\equiv 0$ .

Exact solution  $\psi_\lambda(q)$ , recessive for  $q \rightarrow +\infty$  (WHL specification).

$$\psi_\lambda(q) \sim \Pi_\lambda(q)^{-1/2} \exp \int_q^{+\infty} \Pi_\lambda(\tilde{q}) d\tilde{q}, \quad \Pi_\lambda(q) \stackrel{\text{def}}{=} (V(q) + \lambda)^{1/2} \quad (\text{classical momentum})$$

- Adjacent conjugate solution, recessive for  $q \rightarrow +e^{-i\varphi/2}\infty$ :

$$\Psi_\lambda(q) \stackrel{\text{def}}{=} \psi_{\lambda^{[1]}}^{[1]}(e^{i\varphi/2} q)$$

- $q \rightarrow +\infty$  expansions of  $\psi_\lambda$  and  $\Psi_\lambda$  **fully known**, e.g.:

$$\psi_\lambda(q) \sim e^{\mathcal{C}} q^{-N/4 - \beta_{-1}(\vec{v})} \exp \left\{ - \sum_{\{\sigma > 0\}} \beta_{\sigma-1}(\vec{v}) \frac{q^\sigma}{\sigma} \right\}, \quad \mathcal{C} = \frac{1}{N} \left[ -2 \log 2 \beta_{-1}(\cdot) + \partial_s \frac{\beta_{-1}(\cdot)}{1-2s} \right]_{s=0}$$

$\vdots$

$\implies$  Wronskian **explicitly** evaluates (in  $q \rightarrow +\infty$  limit):

$$\boxed{\psi'_\lambda(q)\Psi_\lambda(q) - \Psi'_\lambda(q)\psi_\lambda(q) \equiv 2i e^{i\varphi/4} e^{i\varphi\beta_{-1}(\vec{v})/2}}$$

- Add **basic exact identities**:

$$D^+(\lambda) \equiv -\psi'_\lambda(0) \quad \Downarrow \quad D^-(\lambda) \equiv \psi_\lambda(0)$$



$$\boxed{-e^{-i\varphi/4} D(\lambda | \mathcal{E}_+) D(e^{-i\varphi} \lambda | \mathcal{E}_+^{[1]}) + e^{+i\varphi/4} D(e^{-i\varphi} \lambda | \mathcal{E}_+^{[1]}) D(\lambda | \mathcal{E}_-) \equiv 2i e^{+i\varphi\beta_{-1}(\vec{v})/2}}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\,\textcolor{red}{D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_+^{[1]})}\,D(\lambda\mid \mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\,\textcolor{red}{D(\lambda\mid \mathcal{E}_+)}\,D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_-^{[1]}) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\,D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_+^{[1]})\,D(\lambda\mid \mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\,D(\lambda\mid \mathcal{E}_+)\,D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_-^{[1]}) \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\,D(\lambda\mid \mathcal{E}_+)\,D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_-^{[-1]}) - \mathrm{e}^{-\mathrm{i}\varphi/4}\,D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_+^{[-1]})\,D(\lambda\mid \mathcal{E}_-) \equiv 2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\text{FOR OVER SPONTANEOUS } \mathcal{E}_+ \text{ AND } \Delta_{2n}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\,D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid\mathcal{E}_{+}^{[1]})\,D(\lambda\mid\mathcal{E}_{-})-\mathrm{e}^{-\mathrm{i}\varphi/4}\,D(\lambda\mid\mathcal{E}_{+})\,D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid\mathcal{E}_{-}^{[1]})\equiv2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}\\ \mathrm{e}^{+\mathrm{i}\varphi/4}\,D(\lambda\mid\mathcal{E}_{+})\,D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda\mid\mathcal{E}_{-}^{[-1]})-\mathrm{e}^{-\mathrm{i}\varphi/4}\,D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda\mid\mathcal{E}_{+}^{[-1]})\,D(\lambda\mid\mathcal{E}_{-})\equiv2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\text{for even spectrum } \mathcal{E}_+ \text{, } \Delta_{2n}(\lambda,\cdot) = D(\lambda\mid \mathcal{E}_+) \text{ .}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\,D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid\mathcal{E}_+^{[1]})\,D(\lambda\mid\mathcal{E}_-) - \mathrm{e}^{-\mathrm{i}\varphi/4}\,\boxed{D(\lambda\mid\mathcal{E}_+)\;D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid\mathcal{E}_-^{[1]})} \equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\,\boxed{D(\lambda\mid\mathcal{E}_+)\;D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda\mid\mathcal{E}_-^{[-1]})} - \mathrm{e}^{-\mathrm{i}\varphi/4}\,D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda\mid\mathcal{E}_+^{[-1]})\,D(\lambda\mid\mathcal{E}_-) \equiv 2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\,D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_+^{[1]})\,D(\lambda\mid \mathcal{E}_-)-\mathrm{e}^{-\mathrm{i}\varphi/4}\,D(\lambda\mid \mathcal{E}_+)\,D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_-^{[1]})\,\equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\,D(\lambda\mid \mathcal{E}_+)\,D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_-^{[-1]})\,-\mathrm{e}^{-\mathrm{i}\varphi/4}\,D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_+^{[-1]})\,D(\lambda\mid \mathcal{E}_-)\equiv 2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\,\textcolor{red}{D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_+^{[1]})}\,D(\lambda\mid \mathcal{E}_-)-\mathrm{e}^{-\mathrm{i}\varphi/4}\,\textcolor{red}{D(\lambda\mid \mathcal{E}_+)\,D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_+^{[1]})}\,\equiv 2\,\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\mathrm{e}^{+\mathrm{i}\varphi/4}\,\textcolor{red}{D(\lambda\mid \mathcal{E}_+)\,D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_-^{[-1]})}-\mathrm{e}^{-\mathrm{i}\varphi/4}\,\textcolor{red}{D(\mathrm{e}^{\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_+^{[-1]})}\,D(\lambda\mid \mathcal{E}_-)\equiv 2\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi\beta_{-1}(\vec{v})/2}$$

$$\implies \quad \frac{D(\mathrm{e}^{-\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_+^{[+1]})}{D(\mathrm{e}^{+\mathrm{i}\varphi}\,\lambda\mid \mathcal{E}_+^{[-1]})}\,=\,-\,\mathrm{e}^{\mathrm{i}\,[\textcolor{red}{-\varphi/2}+\varphi\beta_{-1}(\vec{v})]}\qquad \Bigl(\varphi\stackrel{\mathrm{def}}{=}\frac{4\pi}{N+2}\Bigr)$$

$$2\arg \textcolor{red}{D(-\mathrm{e}^{-\mathrm{i}\varphi}\,E\mid \mathcal{E}_+^{[+1]})}-\varphi\,\beta_{-1}(\vec{v})\,=\,\pi\bigg[k+\frac{1}{2}\textcolor{red}{+}\frac{N-2}{2(N+2)}\bigg]\qquad \text{for }k=\textcolor{red}{2n}\geq 0$$

$$\text{for odd spectrum } \mathcal{E}_{\pm}.$$

$$e^{+i\varphi/4} D(e^{-i\varphi} \lambda | \mathcal{E}_+^{[1]}) D(\lambda | \mathcal{E}_-) - e^{-i\varphi/4} D(\lambda | \mathcal{E}_+) D(e^{-i\varphi} \lambda | \mathcal{E}_-^{[1]}) \equiv 2i e^{+i\varphi\beta_{-1}(\vec{v})/2}$$

$$e^{+i\varphi/4} D(\lambda | \mathcal{E}_+) D(e^{i\varphi} \lambda | \mathcal{E}_-^{[-1]}) - e^{-i\varphi/4} D(e^{i\varphi} \lambda | \mathcal{E}_+^{[-1]}) D(\lambda | \mathcal{E}_-) \equiv 2i e^{-i\varphi\beta_{-1}(\vec{v})/2}$$

$$\implies \frac{D(e^{-i\varphi} \lambda | \mathcal{E}_-^{[+1]})}{D(e^{+i\varphi} \lambda | \mathcal{E}_-^{[-1]})} = -e^{i[+\varphi/2 + \varphi\beta_{-1}(\vec{v})]} \quad (\varphi \stackrel{\text{def}}{=} \frac{4\pi}{N+2})$$

$$2 \arg D(-e^{-i\varphi} E | \mathcal{E}_+^{[+1]}) - \varphi \beta_{-1}(\vec{v}) = \pi \left[ k + \frac{1}{2} \pm \frac{N-2}{2(N+2)} \right] \quad \text{for } k = 2n+1 > 0$$

**Complete set of exact quantization conditions**  
 (for all conjugate,  $\begin{array}{c} \text{even} \\ \text{odd} \end{array}$  spectra  $\mathcal{E}_{\pm}^{[\ell]}$ )

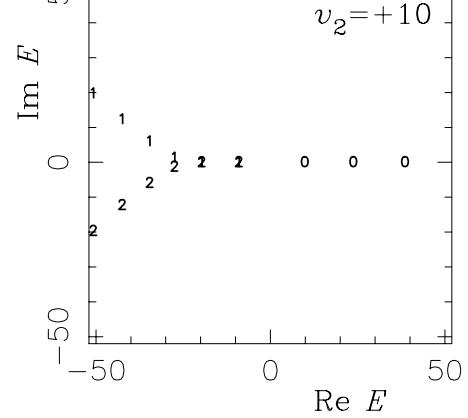
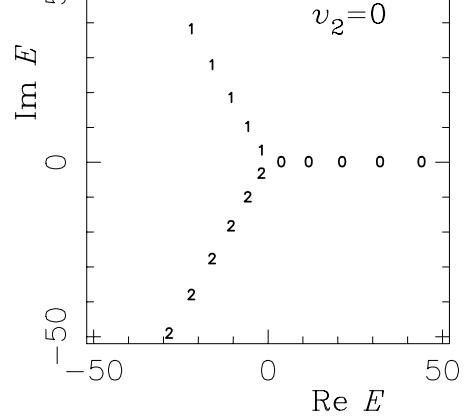
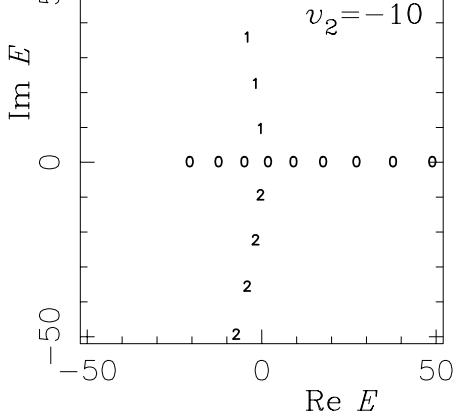
$$\begin{aligned} \frac{1}{i} \left[ \log D(-e^{-i\varphi} E | \mathcal{E}_{\pm}^{[\ell+1]}) - \log D(-e^{+i\varphi} E | \mathcal{E}_{\pm}^{[\ell-1]}) \right] - (-1)^\ell \varphi \beta_{-1}(\vec{v}) \\ = \pi \left[ k + \frac{1}{2} \pm \frac{N-2}{2(N+2)} \right] \quad \text{for } k = \begin{array}{c} 0, 2, 4, \dots \\ 1, 3, 5, \dots \end{array} \quad \ell = 0, 1, \dots, L-1 \pmod{L} \end{aligned}$$

+ structure formulae:

$$\begin{aligned} \log D(\lambda | \mathcal{E}_{\pm}^{[\ell]}) \equiv \lim_{K \rightarrow +\infty} \left\{ \sum_{k < K} \log(E_k^{[\ell]} + \lambda) + \frac{1}{2} \log(E_K^{[\ell]} + \lambda) \right. \\ \left. - \sum_{\{\alpha > 0\}} \frac{1}{2} b_{\alpha}^{[\ell]} [E_K^{[\ell]}]^{\alpha} (\log E_K^{[\ell]} - 1/\alpha) \right\} \\ (k, K \begin{array}{c} \text{even} \\ \text{odd} \end{array}) \end{aligned}$$

altogether define a formally **complete** set of **fixed-point conditions**

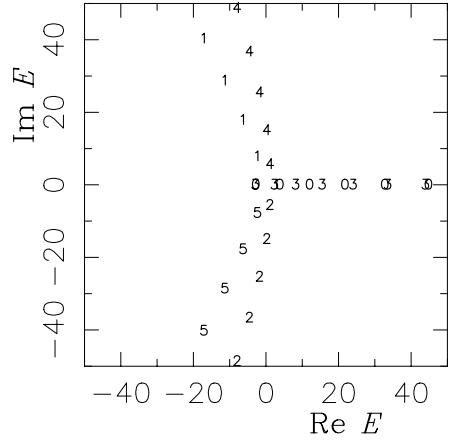
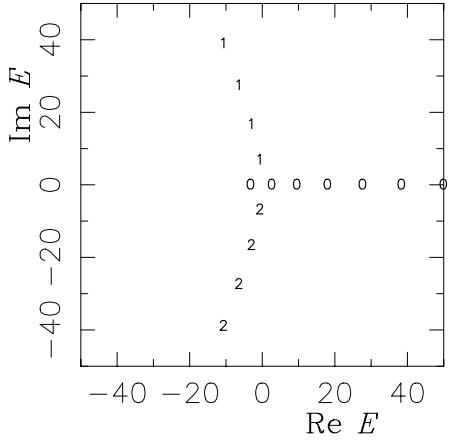
$$(\mathcal{M}^{\pm} \{ \mathcal{E}_{\pm}^{[\ell]} \} = \{ \mathcal{E}_{\pm}^{[\ell]} \} \text{ for some mappings } \mathcal{M}^{\pm}).$$



Same for

$$V(q) = q^4 - 5q^2;$$

$$V(q) \approx q^4 + q^3 - 4.625q^2 - 2.4375q.$$



$$e^{+i\varphi/4} D^+(e^{-i\varphi} \lambda) D^-(\lambda) - e^{-i\varphi/4} D^+(\lambda) D^-(e^{-i\varphi} \lambda) \equiv 2i$$

$$\boxed{\varphi = \frac{4\pi}{N+2}}$$

- Exact quantization condition:

$$\begin{aligned}
2 \Sigma_+(E_k) &= k + \frac{1}{2} + \frac{\kappa}{2} \quad k = 0, 2, 4, \dots \\
2 \Sigma_-(E_k) &= k + \frac{1}{2} - \frac{\kappa}{2} \quad k = 1, 3, 5, \dots \\
\kappa &\stackrel{\text{def}}{=} \frac{N-2}{N+2} \\
\Sigma_{\pm}(E) &\stackrel{\text{def}}{=} \frac{1}{\pi} \sum_{m \text{ even/odd}} \underbrace{\arg(E_m - e^{-i\varphi} E)}_{\phi_m(E)} \quad (N > 2)
\end{aligned}$$

+ boundary condition  $b_\mu E_k^\mu \sim k + \frac{1}{2}$  for  $k \rightarrow +\infty$

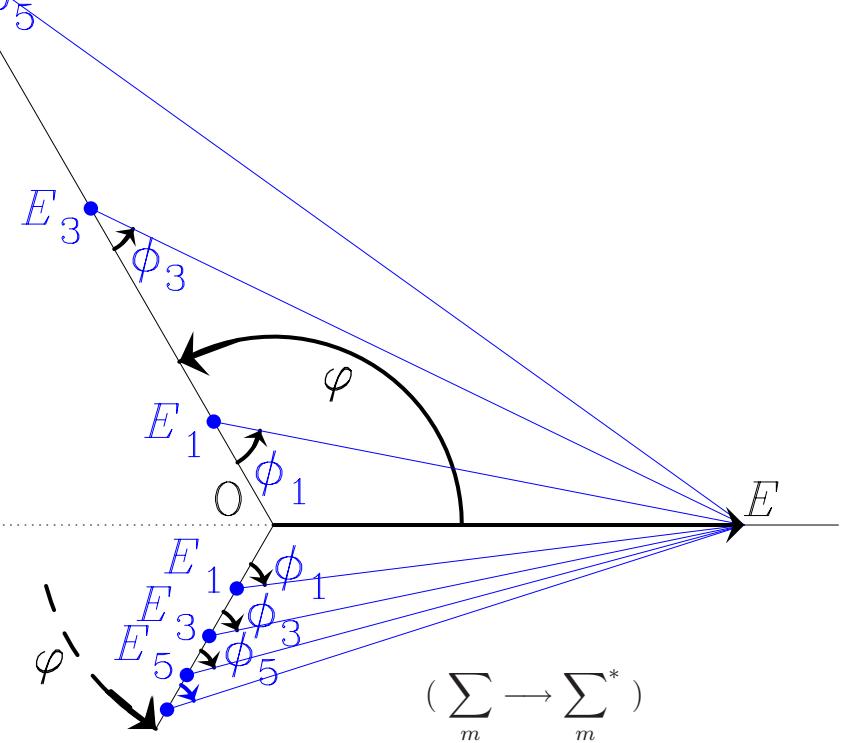
$\iff$  fixed-point equations  $\mathcal{M}^{\pm}\{\mathcal{E}_{\pm}\} = \{\mathcal{E}_{\pm}\}$   
 (mappings  $\mathcal{M}^{\pm}$  proved **globally contractive** for  $N > 2$ , by A. Avila).

•  $N = 4$

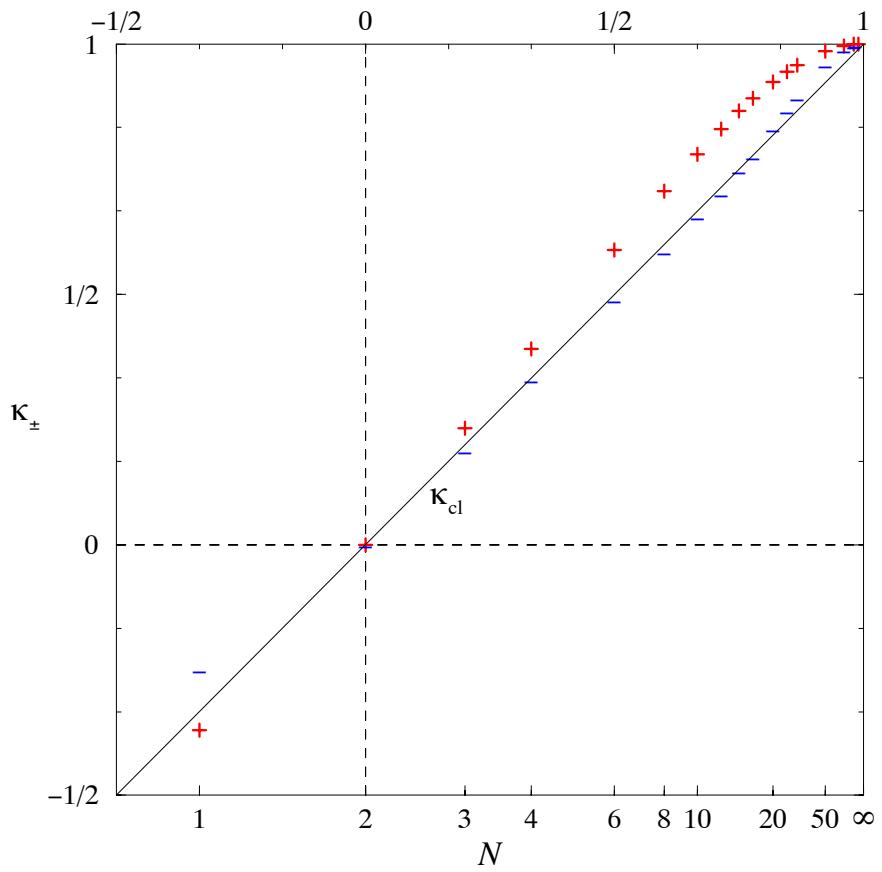
$\kappa = +1/3$

$\kappa = -1/3$

•  $W = J$



Horizontal scale  $\kappa_{\text{cl}} = \kappa \stackrel{\text{def}}{=} \frac{N-2}{N+2}$        $\kappa_+(400) \approx 0.99975$



$$\left(-\frac{d^2}{dq^2} + [V(q) + \lambda]\right) \psi = 0$$

and, e.g.,  $\psi = \psi_\lambda(q)$  recessive for  $q \rightarrow +\infty$  (canonical)  $(\lambda : \text{arbitrary, input})$ .

Restrict to half-line  $[q, +\infty)$  ( $q : \text{parameter}$ ).

Translated **basic identities**:

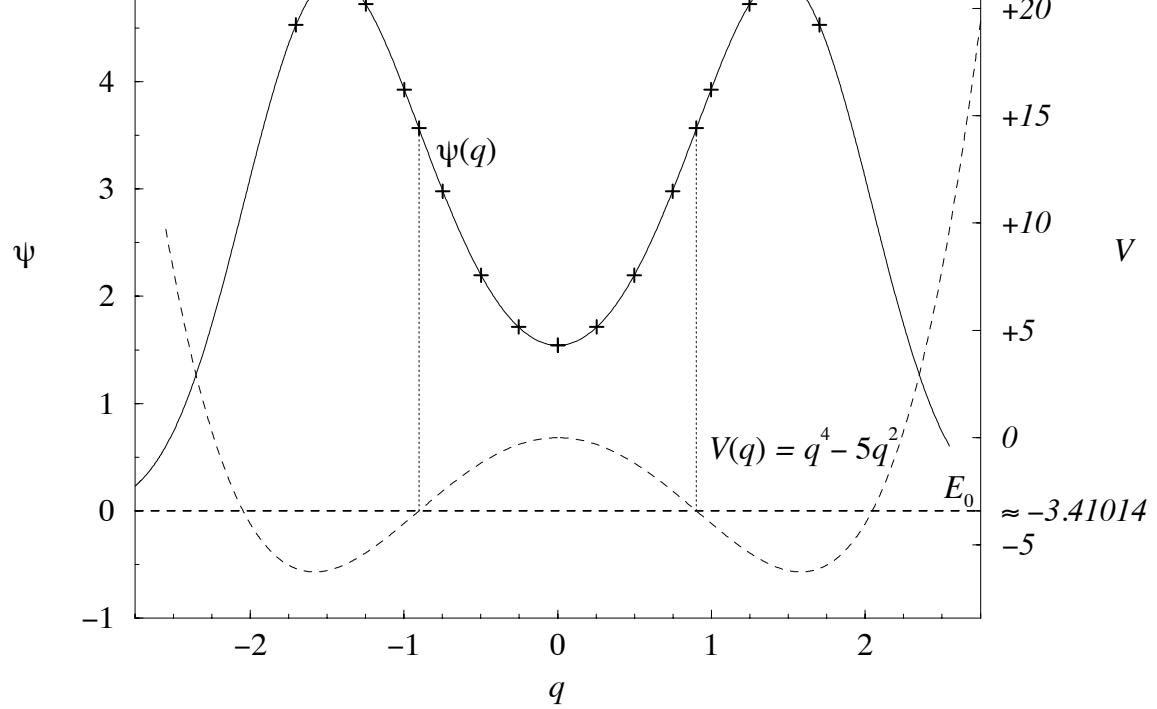
$$\psi_\lambda(q) \equiv D_q^-(\lambda), \quad \psi'_\lambda(q) \equiv -D_q^+(\lambda)$$

hence  $\psi_\lambda(q)$  follows by solving a parametric fixed-point problem at each  $q$  independently:

$$\mathcal{M}_q^- \{\mathcal{E}_{q,-}^{[\ell]}\} = \{\mathcal{E}_{q,-}^{[\ell]}\}$$

for the **Dirichlet spectrum**  $\mathcal{E}_{q,-}$  on the half-line  $[q, +\infty)$ .

(Likewise for  $\psi'_\lambda(q)$ , using the **Neumann spectrum**  $\mathcal{E}_{q,+}$ .)



Ordinary Differential Equations	$\longleftrightarrow$	Integrable Models
<b>1D Schrödinger equation with homogeneous potential <math>q^{2M}</math></b>		<b>2D 6-vertex model with twist <math>\phi = \pi/(2M + 2)</math></b>
Spectral parameter $\lambda$		Spectral parameter $\nu$
Degree of potential $2M$	$e^{2\pi i/(2M+2)} = -e^{-2i\eta}$	Anisotropy $\eta$
Stokes multiplier $C(\lambda)$		Transfer matrix $T(\nu)$
$D^-(\lambda) = \psi_\lambda(0)$		$Q(\nu)$ operator
Exact quantization conditions		Bethe Ansatz equations

(transposed from Dorey–Dunning–Tateo, *The ODE/IM correspondence* [[hep-th/0703066](#)])

Inhomogeneous polynomial potentials.

- contractivity of fixed-point mapping? (Numerically OK near  $\vec{v} = \vec{0}$ )
- correspondence with integrable models (generalized Bethe Ansatz).

- More general problems:

- rational potentials (e.g., centrifugal term)
- all Heun equations
- higher-order equations/systems, higher-dimensional Schrödinger equations, ...

$$\hat{H}(v) = -d^2/dq^2 + \mathbf{q}^N + \mathbf{v}\mathbf{q}^M \text{ (coupled problem)} \approx v^{2/(M+2)} \left[ -d^2/dq^2 + \mathbf{q}^M + g\mathbf{q}^N \right]$$

$$\hat{H}_0(v) = -d^2/dq^2 + \mathbf{v}\mathbf{q}^M \text{ (uncoupled problem)} \approx v^{2/(M+2)} \left[ -d^2/dq^2 + \mathbf{q}^M \right]$$

hence: relate  $\det^{\pm}(\hat{H}(v) + \lambda)$  to  $\det^{\pm}(\hat{H}_0 + \lambda)$  for  $v \rightarrow +\infty \Leftrightarrow g \rightarrow 0^+$  ?

$g \rightarrow 0$ : a most singular limit! E.g., in exact quantization condition

$$2 \arg D(-e^{-i\varphi} \lambda_k | \mathcal{E}_+^{[+1]}) - \varphi \beta_{-1}(\vec{v}) = \pi \left[ k + \frac{1}{2} + \frac{N-2}{2(N+2)} \right] \quad \text{for } k = 2n,$$

- the degree, hence the angle  $\varphi$  as well, jump ( $N \rightarrow M$ );
- the anomaly type, hence  $\beta_{-1}(\vec{v})$  as well, may jump (e.g.,  $\mathbf{N} \rightarrow \mathbf{A}$  for  $\mathbf{q}^2 + g\mathbf{q}^4$ ).

### Main theoretical estimate

$$\det^{\pm}(\hat{H}(v) + \lambda) \sim \left[ \frac{\det_{\text{cl}}(\hat{H}(v) + \lambda)}{\det_{\text{cl}}(\hat{H}_0(v) + \lambda)} \right]^{1/2} \det^{\pm}(\hat{H}_0(v) + \lambda)$$

### Practical implication

There only remains to compute two improper actions,

$$\left( \frac{1}{2} \log \det_{\text{cl}}(\hat{H}(v) + \lambda) = \right) \int_0^{+\infty} \Pi_\lambda(q, v) dq = \int_0^{+\infty} (\mathbf{q}^N + \mathbf{v}\mathbf{q}^M + \lambda)^{1/2} dq \quad (\text{coupled}),$$

$$\left( \frac{1}{2} \log \det_{\text{cl}}(\hat{H}_0(v) + \lambda) = \right) \int_0^{+\infty} \Pi_{0,\lambda}(q, v) dq = \int_0^{+\infty} (\mathbf{v}\mathbf{q}^M + \lambda)^{1/2} dq \quad (\text{uncoupled}).$$

$$\text{Binomial } \Pi(q)^2 = uq^N + vq^M$$

- **Exact** evaluation of improper action integral:

$$\int_0^{+\infty} (uq^N + vq^M)^{1/2} dq = \frac{\Gamma(\frac{M+2}{2(N-M)}) \Gamma(-\frac{N+2}{2(N-M)})}{(N-M) \Gamma(-1/2)} u^{-\frac{M+2}{2(N-M)}} v^{\frac{N+2}{2(N-M)}}$$

when the RHS factor is finite, i.e., in **N**ormal case:  $\frac{N+2}{2(N-M)} \notin \mathbb{N}$ .

$$\text{Trinomial } \Pi(q)^2 = q^N + vq^M + \lambda$$

- **Asymptotic** evaluation of improper action integral for  $v \rightarrow +\infty$ :

$$\begin{aligned} \int_0^{+\infty} (q^N + vq^M + \lambda)^{1/2} dq &\sim \int_0^{+\infty} (q^N + vq^M)^{1/2} dq \quad \left( = C_{N,M} v^{\frac{N+2}{2(N-M)}} \quad [\mathbf{N}] \right) \\ &+ \int_0^{+\infty} (vq^M + \lambda)^{1/2} dq \quad \left( = C'_M \begin{cases} v^{-\frac{1}{M}} \lambda^{\frac{1}{2} + \frac{1}{M}} & M \neq 2 \\ v^{-\frac{1}{2}} \lambda (1 - \log \lambda) & M = 2 \end{cases} \right) \\ &+ \delta_{M,2} \times \left( C''_N \lambda v^{-\frac{1}{2}} (\log v + 2 \log 2) \right). \end{aligned}$$

- **Exactly computable case:**  $N = 4$  (in complete elliptic integrals,  $k$  = modulus)

$$\begin{aligned} \int_0^{+\infty} (q^4 + vq^2 + \lambda)^{1/2} dq &\equiv \\ &\begin{cases} \frac{1}{3} \lambda^{1/4} [(2\sqrt{\lambda} + v)K(\tilde{k}) - 2vE(\tilde{k})], & \tilde{k} = \frac{(2\sqrt{\lambda} - v)^{1/2}}{2\lambda^{1/4}} \quad (v \leq 2\sqrt{\lambda}) \\ \frac{1}{3}(v + 2\sqrt{\lambda})^{1/2} [2\sqrt{\lambda}K(k) - vE(k)], & k = \left( \frac{v - 2\sqrt{\lambda}}{v + 2\sqrt{\lambda}} \right)^{1/2} \quad (v \geq 2\sqrt{\lambda}) \end{cases} \\ &\sim -\frac{1}{3} v^{3/2} + 0 v^{1/2} \log v + 0 v^{1/2} - \frac{1}{4} \lambda v^{-1/2} [\log(\lambda/v^2) - 4 \log 2 - 1]. \end{aligned}$$

$$\frac{\det(-\mathrm{d}^2/\mathrm{d}q^2 + \textcolor{red}{q}^M + gq^N - E)}{\det(-\mathrm{d}^2/\mathrm{d}q^2 + \textcolor{red}{q}^M - E)} \sim g^{-\frac{4}{N(N+2)}\beta_{-1}(0)} \times$$

$$\exp 2 \int_0^{+\infty} (q^N + vq^M)^{1/2} \mathrm{d}q \times$$

$$\exp \left\{ \delta_{M,2} \frac{1}{N-2} [\log g - N \log 2] E \right\}$$

[A]

$$(\text{with } \int_0^{+\infty} (q^N + vq^M)^{1/2} \mathrm{d}q \propto g^{-(M+2)/2(N-M)}).$$

Basic example:  $N = 4, M = 2$

$$\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \textcolor{red}{q}^2 + gq^4 - E\right) \sim \exp\left\{-\frac{2}{3g} + \left[\frac{1}{2}\log g - 2\log 2\right]E\right\} \underbrace{\det\left(-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \textcolor{red}{q}^2 - E\right)}_{2^{E/2}\sqrt{2\pi}/\Gamma(\frac{1}{2}(1-E))}$$

$$\sum_{k=0}^{\infty} \frac{1}{E_k(g) - E} \sim -\left[\frac{1}{2}\log g - 2\log 2\right] - \frac{1}{2}\left[\log 2 + \psi\left(\frac{1}{2}(1-E)\right)\right]$$

$$Z_g(1) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{E_k(g)} \sim -\frac{1}{2}\log g + \frac{1}{2}(\gamma + 5\log 2) \quad (g \rightarrow 0^+).$$

