Non-hermitian Hamiltonians and the Painlevé IV equation

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Non-Hermitian Operators in Quantum Physics
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Topics in the talk

1. SUSY QM and PHA
   - Factorization method
   - Real k-th order SUSY QM
   - Polynomial Heisenberg algebras (PHA)
2. Solutions to P_{IV} with real parameters
   - Real solutions of P_{IV} with real parameters
   - Complex solutions to P_{IV} with real parameters
   - Non-hermitian Hamiltonians
3. Solutions to P_{IV} with complex parameters
   - Complex SUSY QM
   - Non-hermitian Hamiltonians

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Introduction

$$H\psi = E\psi. \tag{1}$$

- In quantum mechanics one must solve an eigenvalue problem to describe a stationary system.
- This involves solving a second order differential equation with boundary conditions.

How do we solve it?

One elegant procedure consists in using the factorization method.

One factorizes a Hamiltonian into first-order differential operators. A generalization of the method gives rise to new solvable Hamiltonians.

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Real k-th order SUSY QM

One starts from a given solvable Hamiltonian

$$H_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0(x), \tag{2}$$

and generates a chain of intertwining relations

$$H_j A_j^+ = A_j^+ H_{j-1}, \quad H_{j-1} A_j^- = A_j^- H_j, \tag{3}$$

$$H_j = -\frac{1}{2}\frac{d^2}{dx^2} + V_j(x), \tag{4}$$

$$A_j^\pm = \frac{1}{\sqrt{2}}\left[\mp\frac{d}{dx} + \alpha_j(x,\epsilon_j)\right], \quad j = 1, \ldots, k. \tag{5}$$

Hence, the following equations must be satisfied

$$\alpha_j'(x,\epsilon_j) + \alpha_j^2(x,\epsilon_j) = 2[V_{j-1}(x) - \epsilon_j], \tag{6}$$

$$V_j(x) = V_{j-1}(x) - \alpha_j'(x,\epsilon_j). \tag{7}$$

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Polynomial Heisenberg algebras (PHA)

Second-order PHA

$$[H, L^{\pm}] = \pm L^{\pm}, \tag{8}$$

$$[L^{-}, L^{+}] \equiv Q_3(H+1) - Q_3(H) = P_2(H). \tag{9}$$

Closed chain

$$L^{-} = L_3^{-} L_2^{-} L_1^{-}, \tag{10}$$

$$L_i^{-} = \frac{1}{\sqrt{2}}(-\partial - f_i), \ \ i = 1, 2, 3, \tag{11}$$

$$H_{i+1} L_i^{+} = L_i^{+} H_i, \tag{12}$$

$$H_i L_i^{-} = L_i^{-} H_{i+1}, \tag{13}$$

We get two different factorizations of the same Hamiltonians plus the closure condition

$$H_{i+1} = L_i^{+} L_i^{-} + \mathcal{E}_i = L_{i+1}^{-} L_{i+1}^{+} + \mathcal{E}_{i+1}. \quad i = 1, 2. \tag{14}$$

$$H_4 = L_3^{+} L_3^{-} + \mathcal{E}_3 = H_1 - 1 = L_1^{-} L_1^{+} + \mathcal{E}_1 - 1. \tag{15}$$

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Closed chain diagram

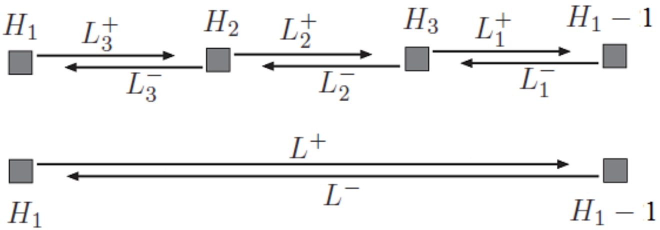

Figure: Diagram of the two equivalent SUSY transformation. Above, the three step first-order operators; below, the one third-order operator.

Painlevé IV equation (P_{IV})

Solving the system of three differential equations and defining $g \equiv f_1 - x$ one gets

$$\boxed{gg'' = \frac{1}{2}(g')^2 + \frac{3}{2}g^4 + 4g^3x + 2g^2\left(x^2 - a\right) + b,} \qquad (16)$$

which is the Painlevé IV equation (P_{IV}) with parameters

$$a = \frac{\mathcal{E}_2 + \mathcal{E}_3}{2} - \mathcal{E}_1 - 1, \quad b = -\frac{(\mathcal{E}_3 - \mathcal{E}_2)^2}{2}. \qquad (17)$$

In general $g \in \mathbb{C}$. Besides, $\mathcal{E}_i \in \mathbb{C}$ which implies that $a, b \in \mathbb{C}$ and so g is a complex solution to P_{IV} associated with the complex parameters a, b. More on this in the third section.

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Real solutions of P_{IV} with real parameters

The first-order SUSY partner of the harmonic oscillator have second-order PHA.

$$u(x;\epsilon) = e^{-x^2/2}\left[{}_1F_1\left(\frac{1-2\epsilon}{4},\frac{1}{2};x^2\right) + 2x\nu\frac{\Gamma\left(\frac{3-2\epsilon}{4}\right)}{\Gamma\left(\frac{1-2\epsilon}{4}\right)}{}_1F_1\left(\frac{3-2\epsilon}{4},\frac{3}{2};x^2\right)\right],$$

$$g(x;\epsilon_1) = -x - \{\ln[\psi_{\mathcal{E}_1}(x)]\}'. \tag{18}$$

The energies of the extremal states of H_1 are

$$\mathcal{E}_1 = \epsilon_1, \quad \mathcal{E}_2 = \frac{1}{2}, \quad \mathcal{E}_3 = \epsilon_1 + 1. \tag{19}$$

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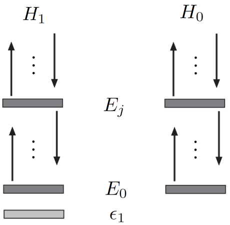
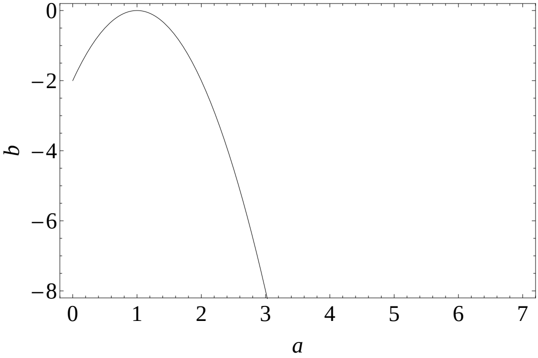
Energy spectrum and solution parameter space (a, b)





Solutions to P_{IV} through higher-order SUSY

There is a theorem stating the conditions for the hermitian higher-order SUSY partners of the harmonic oscillator to be reducible to the second-order PHA.

The main requirement is for the transformation functions

$$u_j = (a^-)^{j-1} u_1, \tag{20}$$

$$\epsilon_j = \epsilon_1 - (j-1), \quad j = 1, \ldots, k, \tag{21}$$

where a^- is the standard annihilation operator of H_0 so that the only free seed is u_1 without roots, associated to a real factorization energy ϵ_1 such that $\epsilon_1 < E_0 = 1/2$. The energies of the extremal states of H_k are

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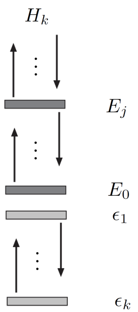
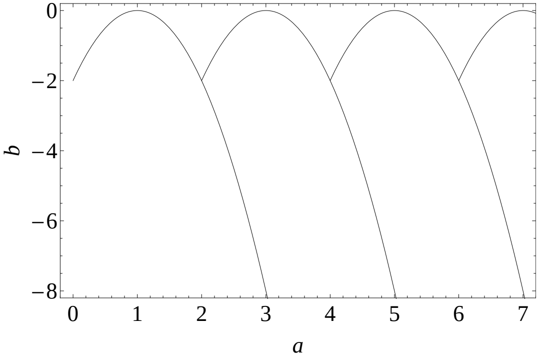
Energy spectra and solution parameter space (a, b)



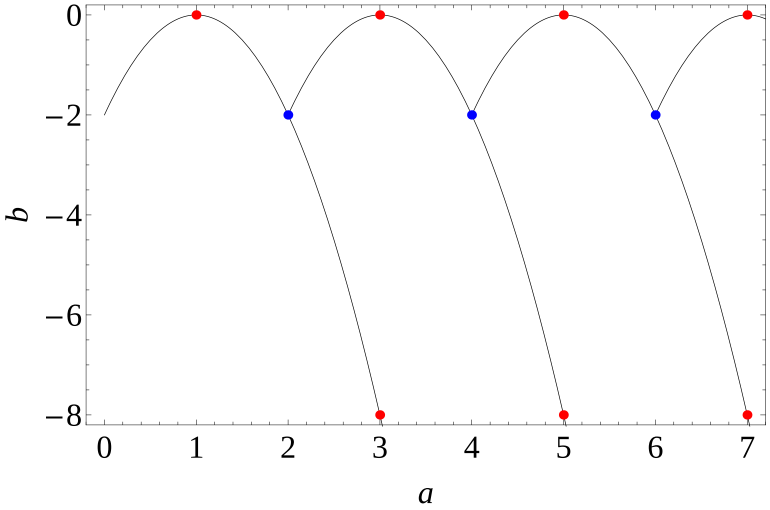


P_{IV} solution hierarchies

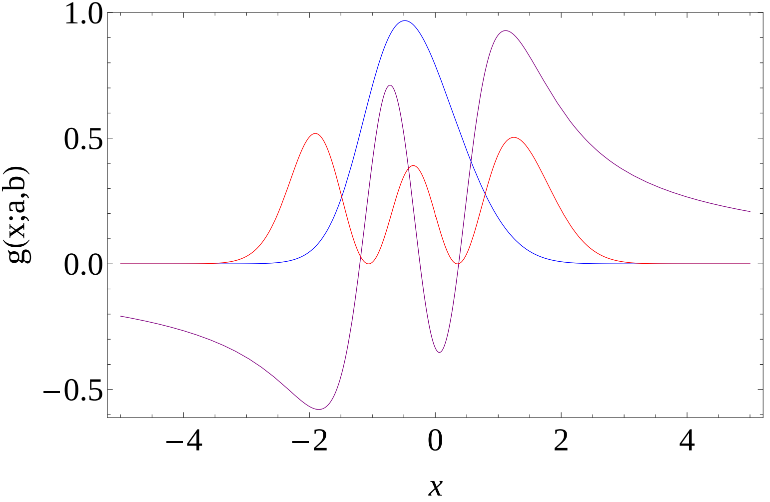
P_{IV} solutions can be classified into three hierarchies according to the functions they depend on as:

- Confluent hypergeometric function (${}_1F_1$) hierarchy.
- Complementary error function (erf) hierarchy.
- Rational hierarchy (P/Q) hierarchy.

Solution parameter space (a, b)



Real solutions to P_{IV}



Complex solution to P_{IV} with real parameters

Now, we intend to overcome the restriction $\epsilon_1 < E_0 = 1/2$, but still obtain non-singular SUSY transformations.

How do we accomplish this?

Using complex SUSY transformations.

And how do we implement them?

The simplest way is with a complex linear combination as

$$u(x;\epsilon) = e^{-x^2/2}\left[{}_1F_1\left(\frac{1-2\epsilon}{4},\frac{1}{2};x^2\right) + x(\lambda + i\kappa)\,{}_1F_1\left(\frac{3-2\epsilon}{4},\frac{3}{2};x^2\right)\right].$$

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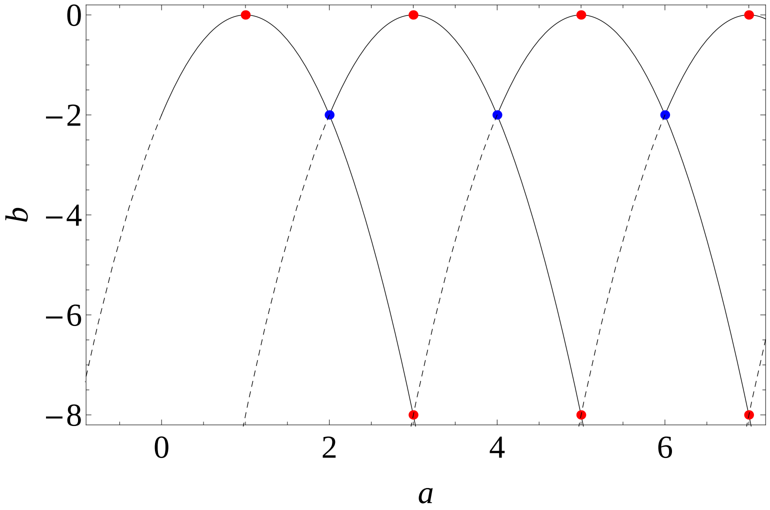
Solution parameter space (a, b)



Expanding the solution families

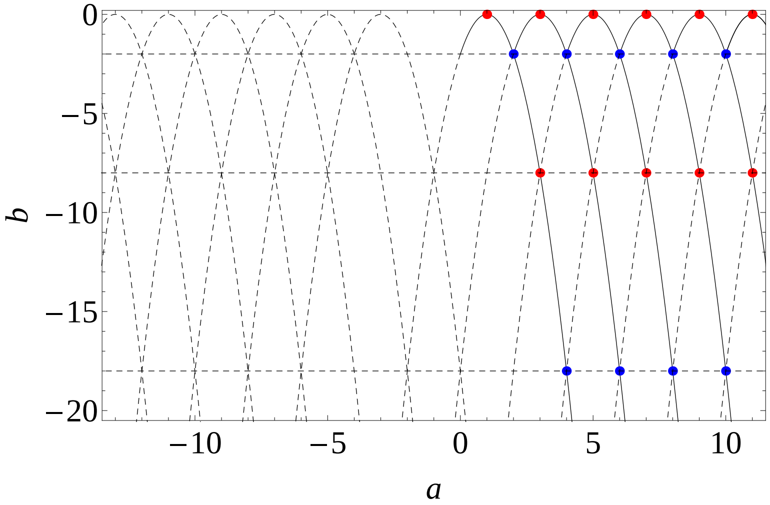
Also note that by making cyclic permutations of the indices of the three energies $\mathcal{E}$ and their extremal states, we expand the solution families to three different sets, defined by

$$a_1 = -\epsilon_1 + 2k - \frac{3}{2}, \quad b_1 = -2\left(\epsilon_1 + \frac{1}{2}\right)^2, \tag{23}$$

$$a_2 = 2\epsilon_1 - k, \quad b_2 = -2k^2, \tag{24}$$

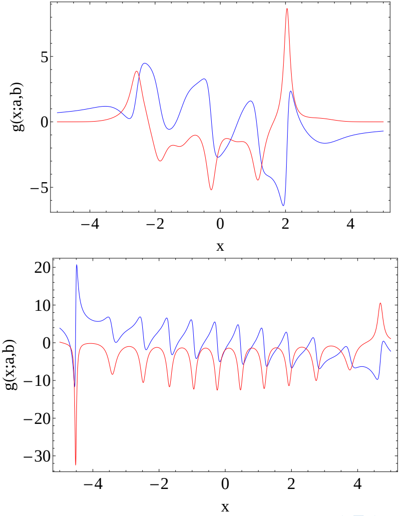
$$a_3 = -\epsilon_1 - k - \frac{3}{2}, \quad b_3 = -2\left(\epsilon_1 - k + \frac{1}{2}\right)^2, \tag{25}$$

Solution parameter space (a, b)



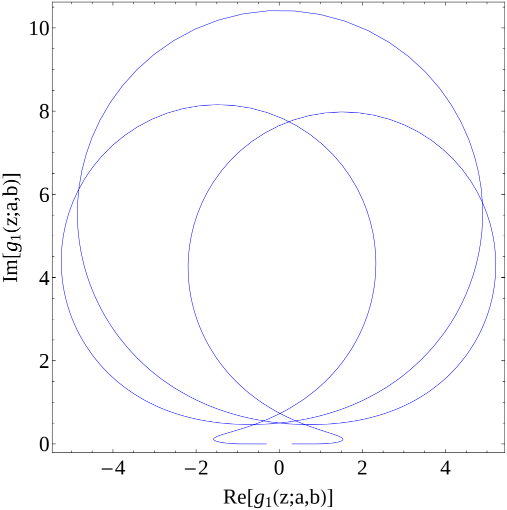
Complex solutions to P_{IV}

Real and imaginary parts



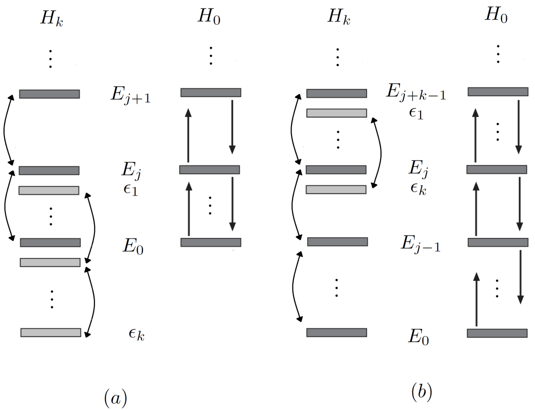
Complex solutions to P_{IV}

Parametric plot



Energy spectrum

Recall that using complex SUSY transformation we have overcomed the restriction $\epsilon_1 < E_0 = 1/2$, then we can have spectra with some or all new levels above the original ground state.



Eigenfunctions

The eigenfunctions of the new non-hermitian Hamiltonian are

$$E_n, \quad \psi_n^{(k)} \propto B_k^+ \psi_n, \tag{26}$$

$$\epsilon_j, \quad \psi_{\epsilon_j}^{(k)} \propto \frac{W(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_k)}{W(u_1, \ldots, u_k)}. \tag{27}$$

which are all square-integrable. The extremal states are

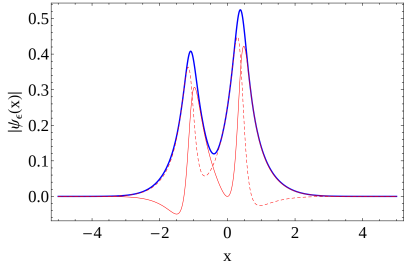
$$\psi_{\mathcal{E}_1} \propto \frac{W(u_1, \ldots, u_{k-1})}{W(u_1, \ldots, u_k)}, \quad \mathcal{E}_1 = \epsilon_k = \epsilon_1 - (k-1), \tag{28}$$

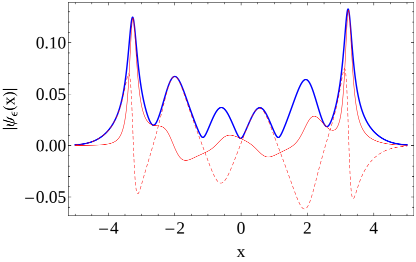
$$\psi_{\mathcal{E}_2} \propto B_k^+ e^{-x^2/2}, \quad \mathcal{E}_2 = \frac{1}{2}, \tag{29}$$

$$\psi_{\mathcal{E}_3} \propto B_k^+ a^+ u_1, \quad \mathcal{E}_3 = \epsilon_1 + 1. \tag{30}$$

the first two can be square integrable.

Eigenfunctions





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More complex solutions to P_{IV}

We have already obtained complex solutions to P_{IV} using complex linear combination of the transformation functions. This lead us to complex solutions associated with real parameters of P_{IV}.

What if we would like to expand the solution domain to complex parameters a, b of P_{IV}?

We can do it using a complex transformation energy ϵ.

And how do we implement them?

Through complex SUSY QM.

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Complex SUSY QM

One starts from a given solvable Hamiltonian

$$H = -\partial^2 + V(x), \tag{31}$$

and propose that H is factorizable as

$$H = A^- A^+ + \epsilon. \tag{32}$$

Usually $A^+ \equiv (A^-)^\dagger$ but here we simply ask that

$$A^+ = -\partial + \beta(x), \tag{33a}$$

$$A^- = \partial + \beta(x), \tag{33b}$$

Now we introduce a new Hamiltonian $\tilde{H} = -\partial^2 + \tilde{V}$ with a reversed factorization

$$\tilde{H} = A^+ A^- + \epsilon, \tag{34}$$

Complex SUSY QM

Hence, the following equations must be satisfied

$$\beta' + \beta^2 = V(x) - \epsilon, \tag{35}$$

$$\tilde{V}(x) = V(x) - 2\beta'(x). \tag{36}$$

and the well known *intertwining relationships*

$$\tilde{H}A^+ = A^+H, \tag{37}$$

$$HA^- = A^-\tilde{H}, \tag{38}$$

$$\tilde{\psi}_k \propto A^+\psi_k(x) \propto \frac{W(u,\psi_k)}{u}, \tag{39}$$

$$\mathrm{Sp}(\tilde{H}) = \{\epsilon\} \cup \{E_n, n = 0, 1, \ldots\}, \tag{40}$$

and $\epsilon \in \mathbb{C}$

Examples of complex potentials

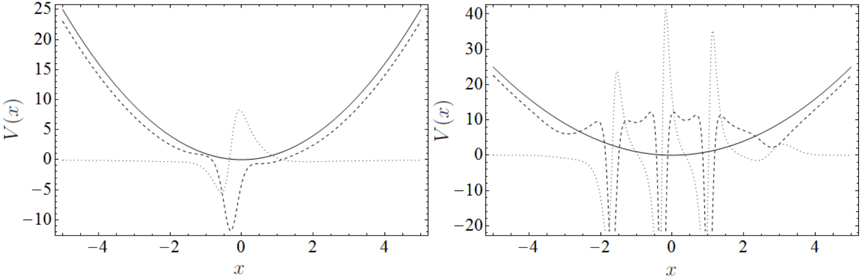

Figure: Examples of SUSY partner potentials of the harmonic oscillator using the two complex factorization energies $\epsilon = -1 + i$ and $\epsilon = 3 + i10^{-3}$. Its real (dashed line) and imaginary (dotted line) parts are compared to the harmonic oscillator (solid line).

Energy spectrum

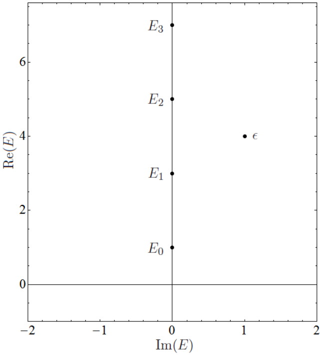

Figure: The complex energy plane which contains the eigenvalues of $\tilde{H}$.

Eigenfunctions

The eigenstates $\tilde{\psi}_k$ are not automatically normalized as in the real SUSY QM since now

$$\langle A^+\psi_n | A^+\psi_n\rangle = \langle \psi_n | (A^+)^\dagger A^+ \psi_n\rangle, \tag{41}$$

and in this case $(A^+)^\dagger A^+ \neq (H - \epsilon)$. Nevertheless, since they are normalizable we can introduce a normalizing constant C_n, chosen for simplicity as $C_n \in \mathbb{R}^+$, so that

$$\tilde{\psi}_n(x) = C_n A^+ \psi_n(x), \quad \langle \tilde{\psi}_n | \tilde{\psi}_n \rangle = 1. \tag{42}$$

Finally, there is a wavefunction that is also eigenfunction of $\tilde{H}$ as $\tilde{H}\tilde{\psi}_\epsilon = \epsilon\tilde{\psi}_\epsilon$:

$$\tilde{\psi}_\epsilon \propto \frac{1}{u}, \tag{43}$$

Eigenfunctions

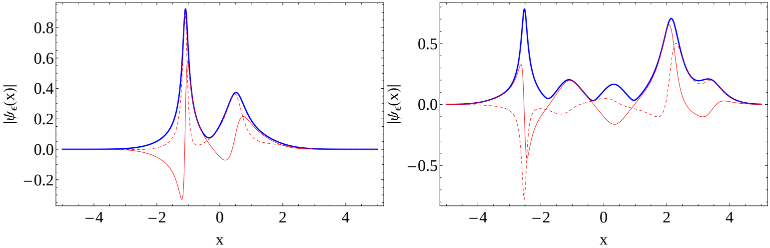

Figure: Wave functions associated to the new complex eigenvalues given by the factorization energies $\epsilon = -1 + i$ and $\epsilon = 3 + i10^{-3}$. The red solid line is the real part, the red dashed line is the imaginary part and the blue line is the absolute value.

Conclusions

- Based on PHA and SUSY QM, we have introduced a method to obtain real and complex solutions to P_{IV} with real parameters.
- Furthermore, we have obtained complex solutions to P_{IV} with complex parameters, thus expanding the solution subspace.
- We have studied the algebras, the eigenfunctions and the spectra of the non-hermitian SUSY generated Hamiltonians.

Future work

- We would like to generalize current formalism to include complex higher-order SUSY transformations and to expand the solution complex space of the parameters of P_{IV}.
- Investigate connection of P_{IV} with other systems ruled by second order PHA and obtain new solutions. Inverse oscillator.

Merci

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http://arxiv.org/abs/1012.0290
http://dx.doi.org/10.3842/SIGMA.2011.025

http://arxiv.org/abs/1104.3599
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