

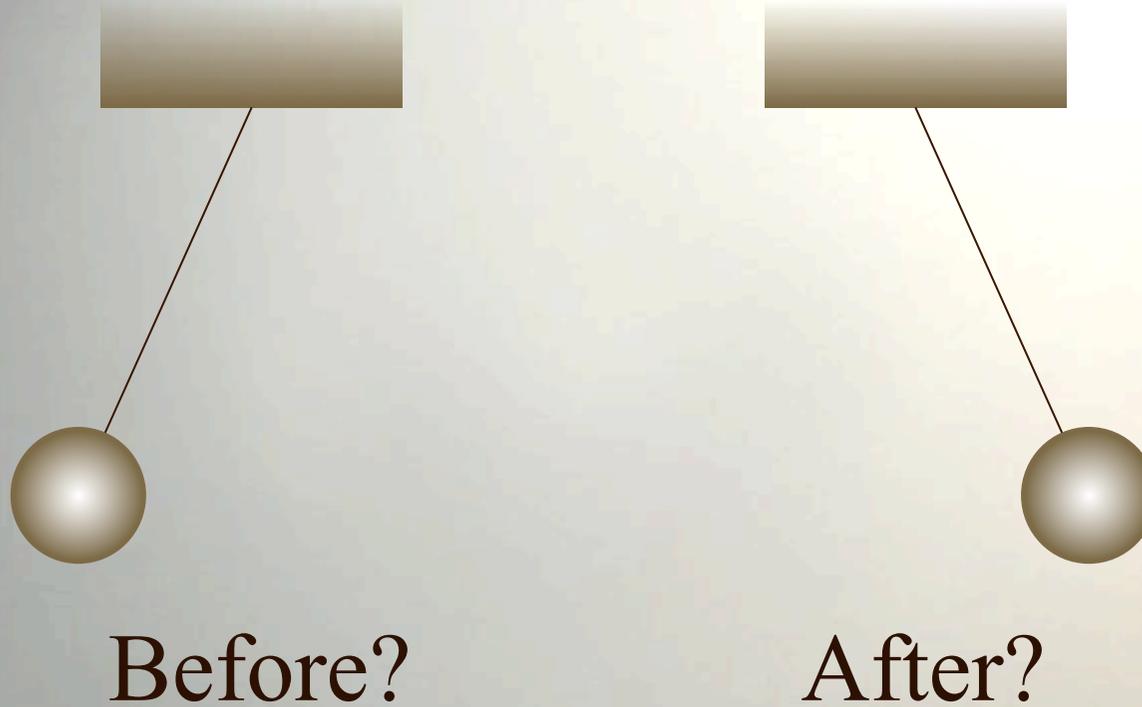
Complex eigenvalue problem of the Hamiltonian and the Liouvillian of an open quantum dot system

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Arrow of Time



Time-reversal symmetry

Arrow of Time



Second law of thermodynamics

Open quantum system

Eq. of motion: Time-reversal symmetry

Its solutions: Can break the symmetry

Open quantum system

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) \psi(x) = E\psi(x)$$

resonance: $E = E_r - i\Gamma/2$

anti-resonance: $E = E_r + i\Gamma/2$

To find the arrow of time

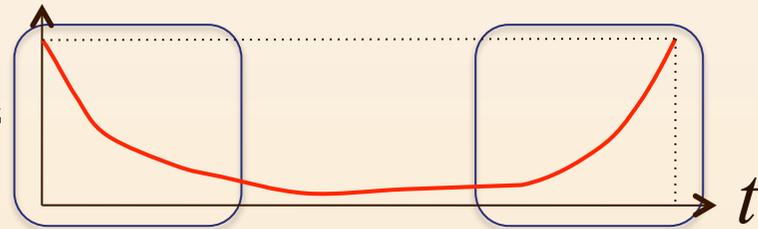
- Finite System + Dissipation added by hand

- Finite System + Infinite Heat Bath

Traces out the heat bath + Markov approx., Perturb.
Time-reversal symmetry is broken somewhere.

- Finite System + Finite System

Everything microscopically.



“We do not live long enough to see the recurrence.”

- Finite System (quantum dot) + Infinite System (lead)

Complex eigenv. prob.: everything microscopically.

Spontaneous breaking of time-reversal symmetry.

Complex eigenvalue problem

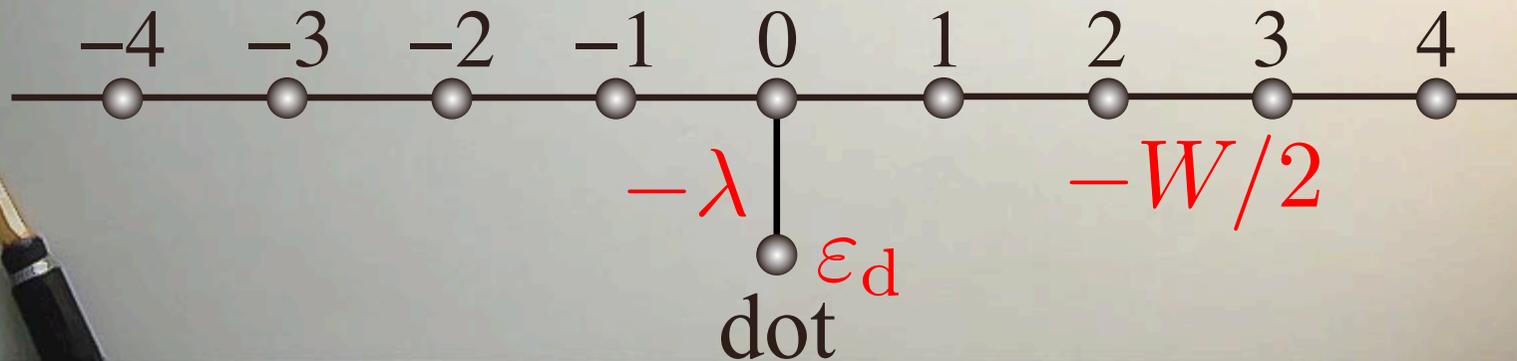
For a simple open quantum dot system, we (numerically) exactly solve the complex eigenvalue problems of

1. the Hamiltonian and
2. the Liouvillian.

Open quantum dot system

Tight-binding model

$$H = -\frac{W}{2} \sum_{-\infty}^{\infty} (|x+1\rangle\langle x| + |x\rangle\langle x+1|) - \lambda (|d\rangle\langle 0| + |0\rangle\langle d|) + \varepsilon_d |d\rangle\langle d|$$

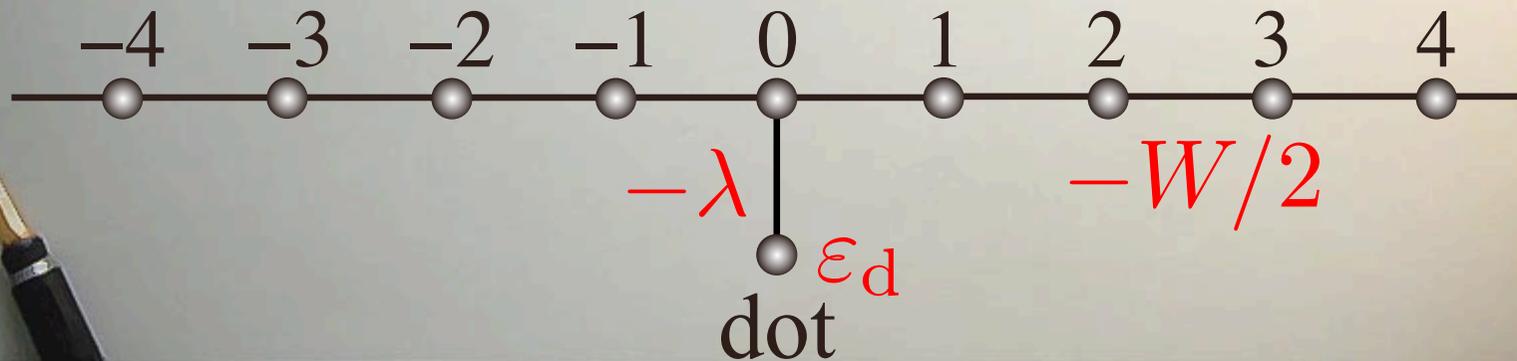


Complex eigenvalues of the Hamiltonian

$$H\psi_n = E_n\psi_n \quad E_n \in \mathbb{C}$$

Two methods

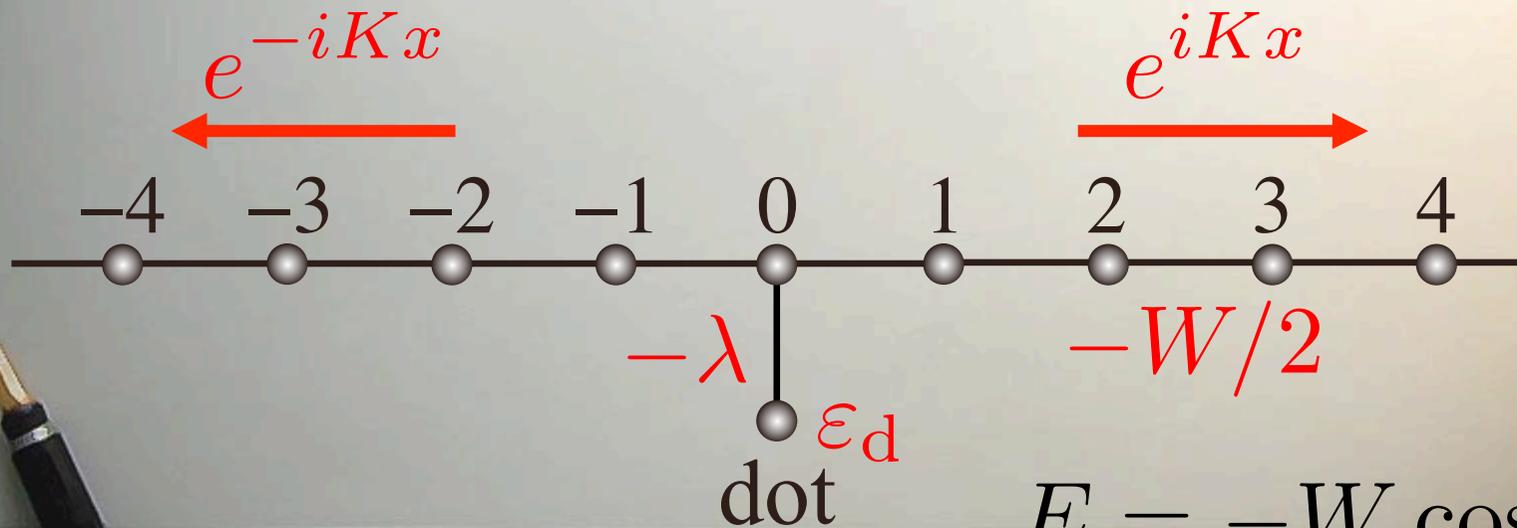
1. Siegert boundary condition
2. Feshbach formalism



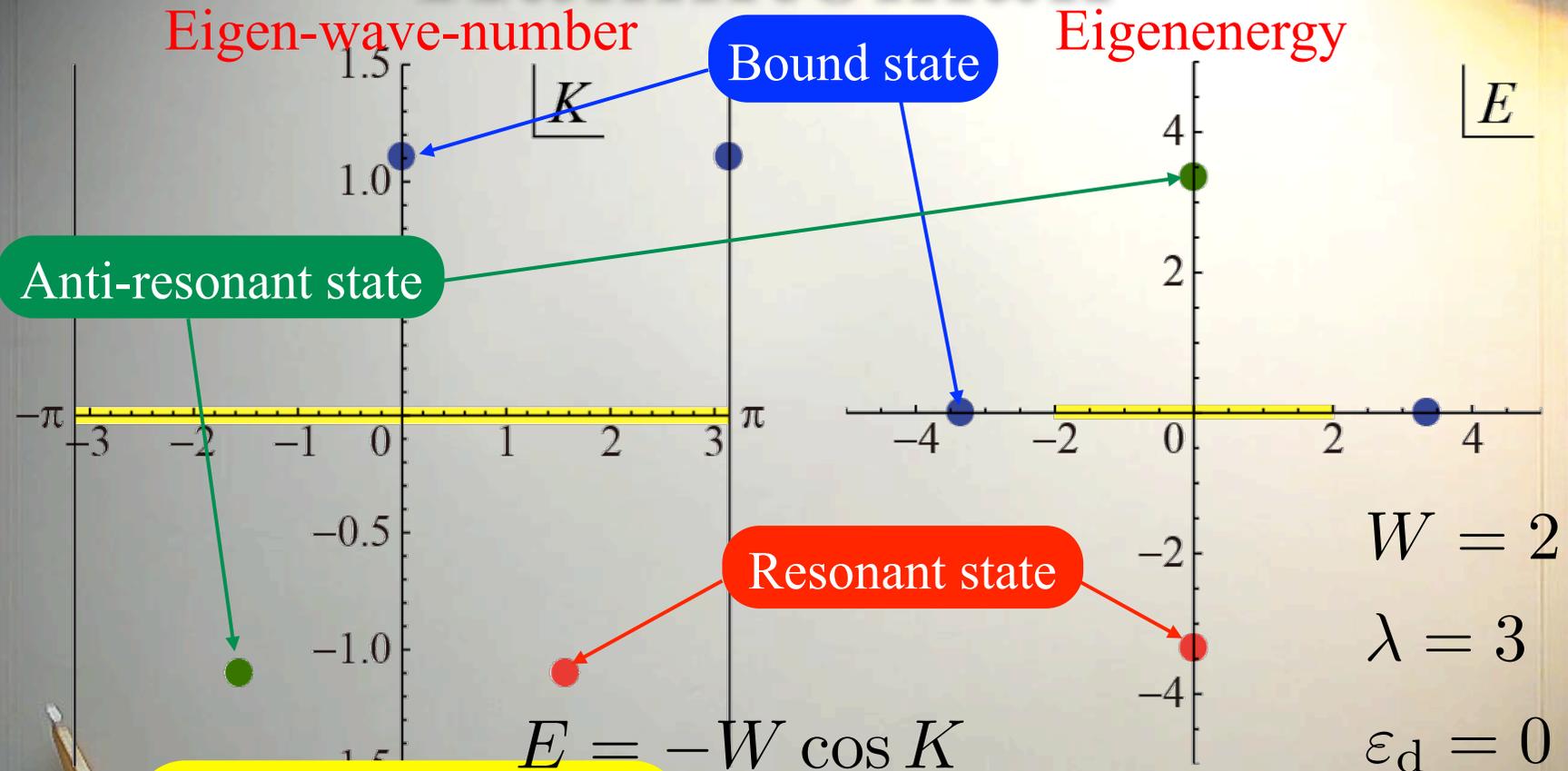
Siegert boundary condition

Resonance:
Eigenstate with outgoing waves only.

$$\psi(x) \approx e^{iK|x|}$$



Complex eigenvalues of the Hamiltonian



$\psi(x) \approx e^{iK|x|}$

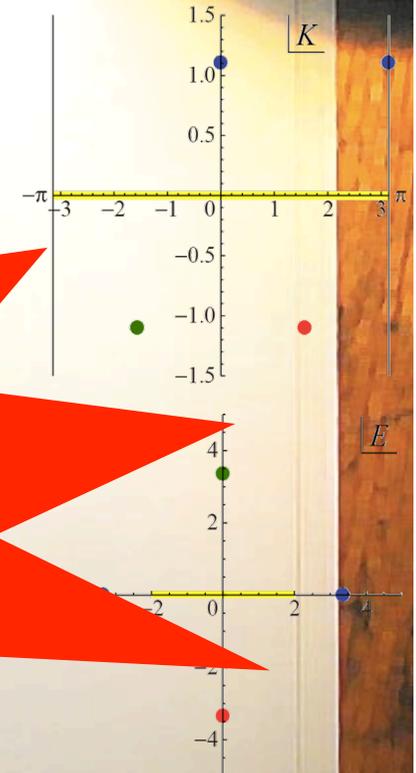
$\text{Re } K_n > 0 \iff \text{Im } E_n < 0$
 $\text{Re } K_n < 0 \iff \text{Im } E_n > 0$

Resonant state as a stationary eigenstate

N. Hatano, K. Sasada, H. Nakamura and T. Petrosky, Prog. Theor. Phys. **119** (2008) 187

$$\langle x | \Psi_n(t) \rangle \approx e^{iK_n x - iE_n t}$$

Breaking of
time-reversal
symmetry



“Resonant state” as an eigenstate

Feshbach Formalism

I. Prigogine and T. Petrosky; I. Rotter et al.

$$H|\psi\rangle = E|\psi\rangle$$

$$P + Q = 1$$

$$H(P + Q)|\psi\rangle = E|\psi\rangle$$

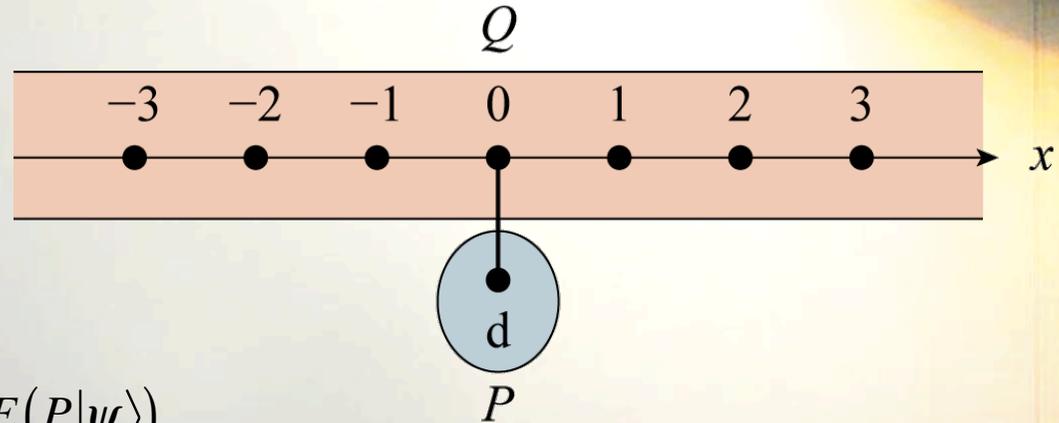
$$\begin{cases} (PHP + PHQ)|\psi\rangle = EP|\psi\rangle \\ (QHP + QHQ)|\psi\rangle = EQ|\psi\rangle \end{cases}$$

$$\begin{cases} PHP(P|\psi\rangle) + PHQ(Q|\psi\rangle) = E(P|\psi\rangle) \\ QHP(P|\psi\rangle) + QHQ(Q|\psi\rangle) = E(Q|\psi\rangle) \end{cases}$$

$$Q|\psi\rangle = \frac{1}{E - QHQ} QHP(P|\psi\rangle)$$

$$\left(PHP + PHQ \frac{1}{E - QHQ} QHP \right) (P|\psi\rangle) = E(P|\psi\rangle)$$

$$H_{\text{eff}}(E)(P|\psi\rangle) = E(P|\psi\rangle)$$



$$\frac{1}{E - QHQ}$$

Retarded \Leftrightarrow Resonance
Advanced \Leftrightarrow Anti-Resonance

Feshbach Formalism

$$H_{\text{eff}}(E) = PHP + PHQ \frac{1}{E - QHQ} QHP$$

$$PHP = 0$$

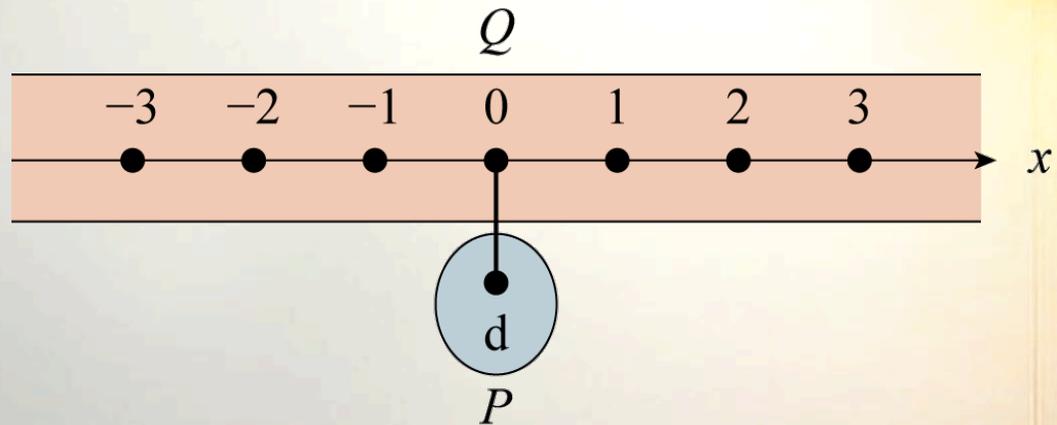
$$QHP = -\lambda|0\rangle\langle d|$$

$$PHQ = -\lambda|d\rangle\langle 0|$$

$$QHQ = -\frac{W}{2} \sum_{x=-\infty}^{\infty} (|x+1\rangle\langle x| + |x\rangle\langle x+1|)$$

$$PHQ \frac{1}{E - QHQ} QHP = \frac{\lambda^2 |d\rangle\langle d|}{E + W e^{iK}}$$

$$E = -W \cos K$$



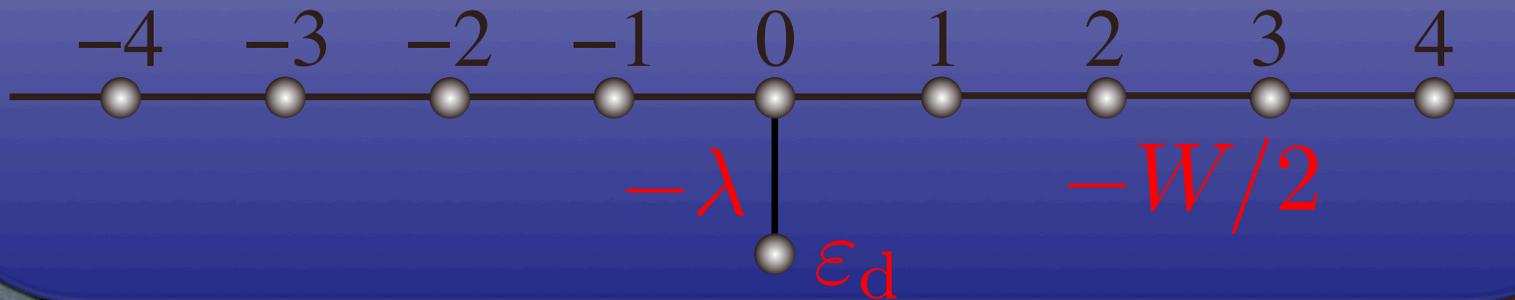
Hidden
non-
Hermiticity

Complex eigenvalues of the Hamiltonian

For small λ , the solutions are

$$E_{\text{bound}} = -W - \frac{8\lambda^4}{W^3}$$

$$E_{\text{res}} = -i \frac{\lambda^2}{W}$$



Approach to Equilibrium

$$\rho(t) \rightarrow \rho_{\text{eq}} \equiv \frac{1}{Z} \sum_n e^{-\beta E_n} |n\rangle\langle n|$$

Mixed State

Liouville-von Neumann equation

$$i\hbar \frac{d\rho}{dt} = [H, \rho] =: L\rho$$

$$L\rho_n = z_n \rho_n \quad z_n \in \mathbb{C}$$

$$\rho(t) \simeq \rho_0 + e^{-iz_1 t} \rho_1 \simeq \rho_0 + e^{-(\text{Im } z_1)t} \rho_1$$

Complex eigenvalues of the Liouvillian

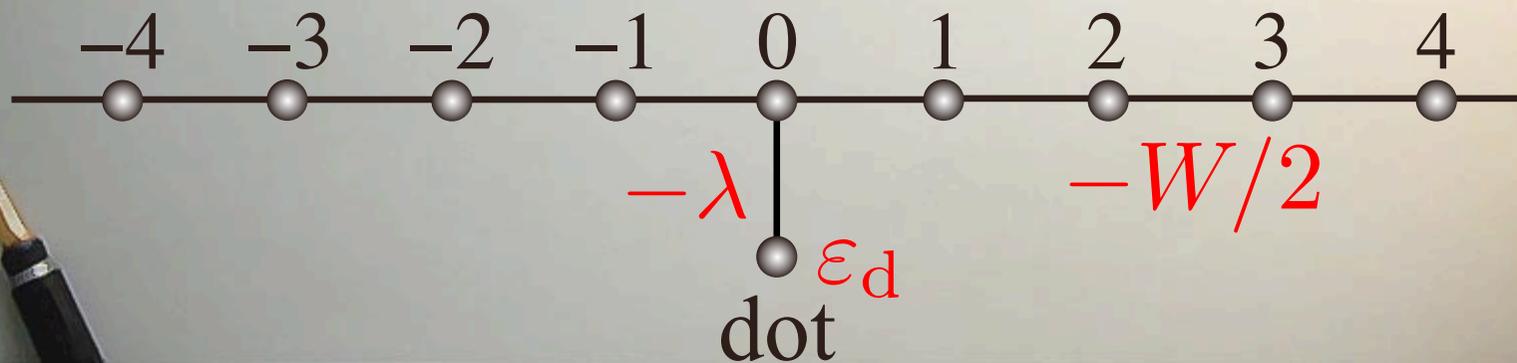
$$L\rho_n = z_n\rho_n \quad z_n \in \mathbb{C}$$

‘Trivial’

$$L|m\rangle\langle n| = [H, |m\rangle\langle n|]$$

‘Nontrivial’

$$= (E_m - E_n)|m\rangle\langle n|$$

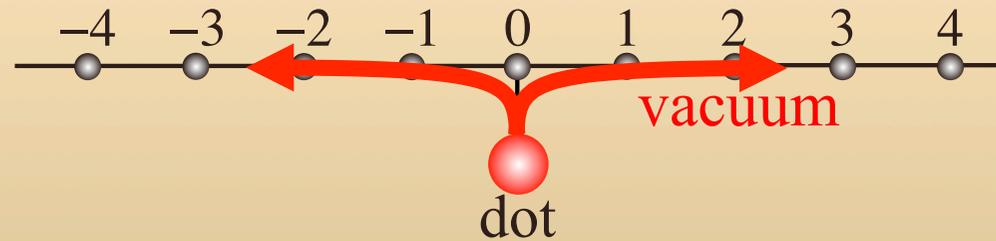


Rate equation

Case 1: Pure-state dynamics
Schrödinger equation

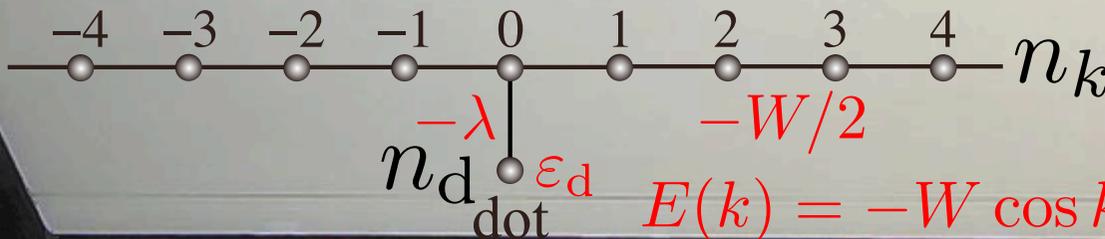
$$\frac{dn_d(t)}{dt}$$

$$\frac{dn_k(t)}{dt}$$



Conventional decay constant

$$n_k = O(1/N) \implies n_d(t) = e^{-2\gamma t} n_d(0)$$



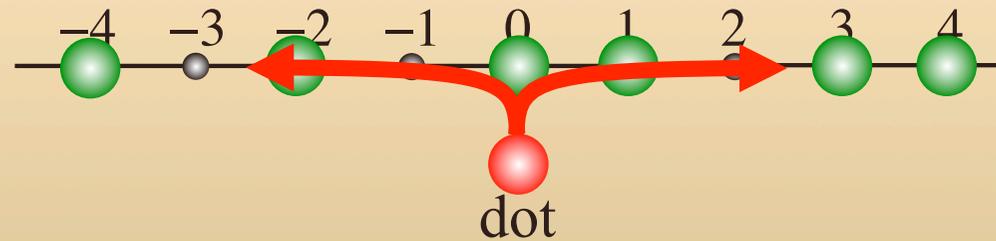
$$\gamma = \frac{\lambda^2}{W}$$

Rate equation

Case 2: Mixed-state dynamics
Liouville-von Neumann equation

$$\frac{dn_d(t)}{dt}$$

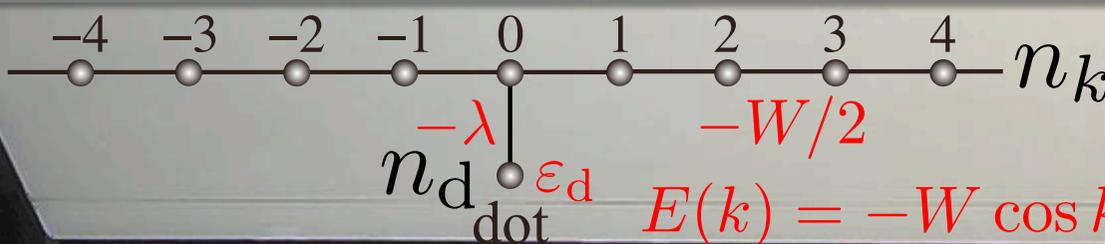
$$\frac{dn_k(t)}{dt}$$



Conventional decay constant

$$n_k =$$

$$n_d(t) - n_d(\infty) = e^{-2\gamma t} (n_d(0) - n_d(\infty))$$



$$\gamma = \frac{\lambda^2}{W}$$

Rate equation

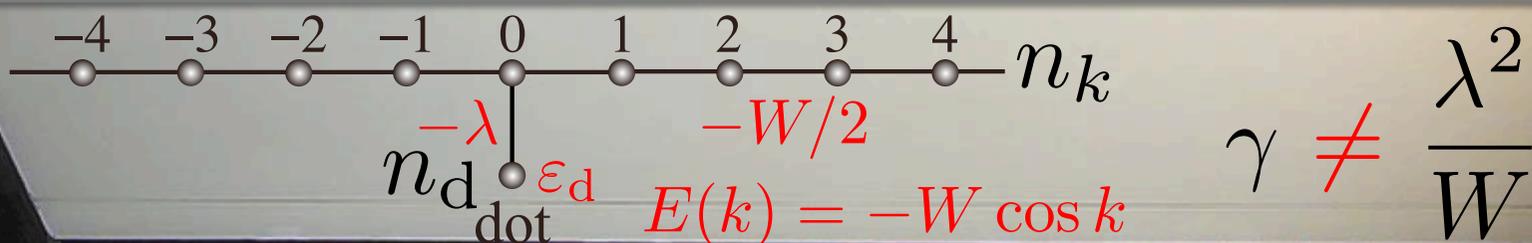
$\lambda^2 t$ approximation

$$\frac{dn_d(t)}{dt} = \frac{\lambda^2}{W^2} \int_{-\pi}^{\pi} \delta(\varepsilon_d - E(k))(n_k(t) - n_d(t)) dk$$

$$\frac{dn_k(t)}{dt} = -\frac{2\pi\lambda^2}{NW^2} \delta(\varepsilon_d - E(k))(n_k(t) - n_d(t))$$

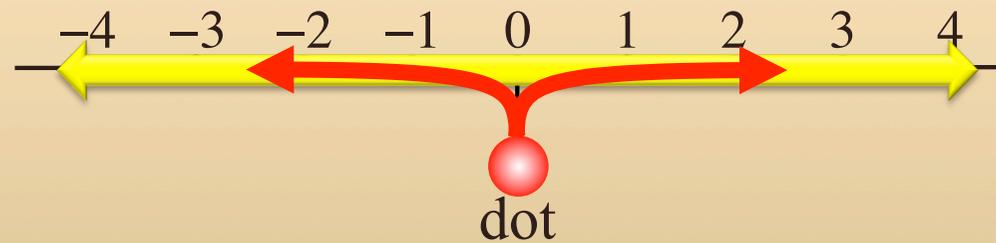
$n_k = O(N)$ for $E(k) = \varepsilon_d$

\implies time dependence of $n_k(t)$



Decay into a laser field

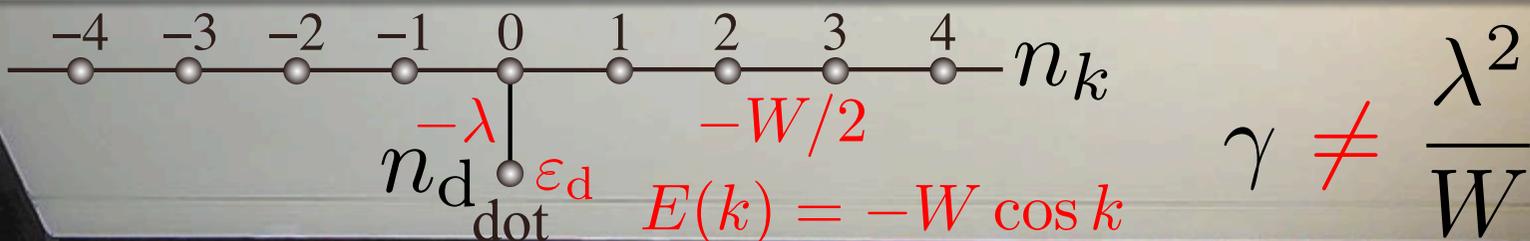
Case 3: Mixed-state dynamics
 Liouville-von Neumann equation



UNconventional decay constant

$$n_k = O(\lambda) \text{ for } E(k) = \epsilon_d$$

\implies time dependence of $n_k(t)$



Bra-Ket notation for the Liouvillian

T. Petrosky, I. Prigogine

$$i\hbar \frac{d\rho}{dt} = [H, \rho] =: L\rho$$

$$H |n\rangle = E_n |n\rangle$$

$$\langle m | n \rangle = \delta_{mn}$$

$$\sum_n |n\rangle \langle n| = 1$$



$$L |m, n\rangle\rangle = \lambda_{mn} |m, n\rangle\rangle$$

$$\langle\langle m, n | k, l \rangle\rangle = \delta_{mk} \delta_{ln}$$

$$\sum_{m,n} |m, n\rangle\rangle \langle\langle m, n| = 1$$

$$|m, n\rangle\rangle := |m\rangle \langle n|$$

$$\lambda_{mn} := E_m - E_n$$

$$\langle\langle A | B \rangle\rangle := \text{Tr } A^\dagger B$$

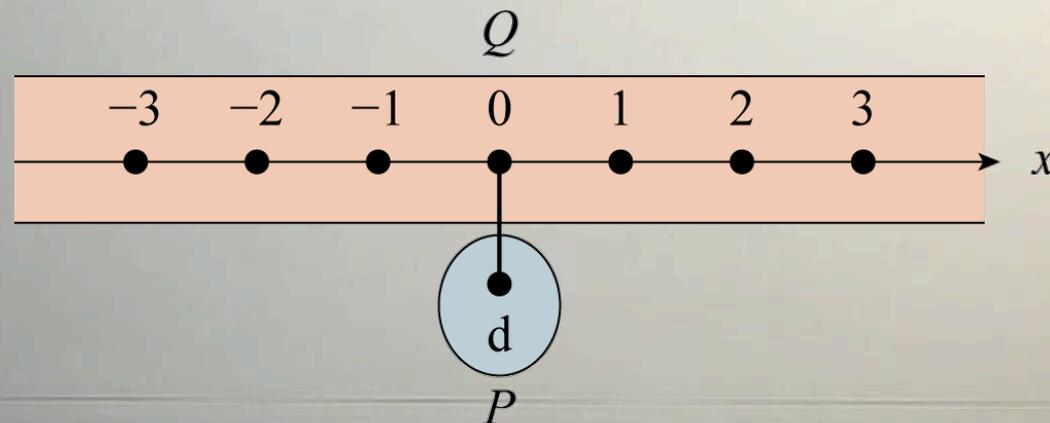
Feshbach formalism for the Liouvillian

R. Nakano, T. Mori, N. Hatano, T. Petrosky

$$L|\rho\rangle\rangle = z|\rho\rangle\rangle$$

$$\left(P_L L P_L + P_L L Q_L \frac{1}{z - Q_L L Q_L} Q_L L P_L \right) (P_L |\rho\rangle\rangle) = z (P_L |\rho\rangle\rangle)$$

$$L_{\text{eff}}(z) (P_L |\rho\rangle\rangle) = z (P_L |\rho\rangle\rangle)$$



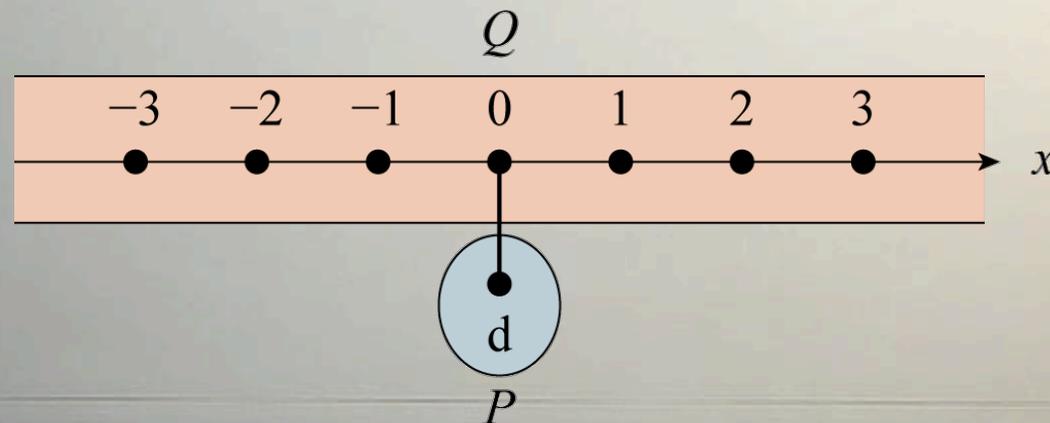
Interaction of ket and bra states

R. Nakano, T. Mori, N. Hatano, T. Petrosky

$$P_L + Q_L = 1 \quad \text{where} \quad P_L = P \times P$$

$$Q_L = Q \times Q + Q \times P + P \times Q$$

Interaction between ket and bra spaces



Retarded and Advanced Functions

R. Nakano, T. Mori, N. Hatano, T. Petrosky

$$\frac{1}{z - L} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\zeta \frac{1}{\zeta + \frac{z}{2} - H} \times \frac{1}{\zeta - \frac{z}{2} - H}$$

4 types of
Green's fns.

=

Ret. or
Adv.

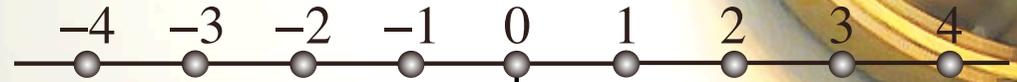
×

Ret. or
Adv.

$$G^{\text{RR}}(z) = (G^{\text{AA}}(z^*))^*$$

$$G^{\text{RA}}(z) = (G^{\text{AR}}(z^*))^*$$

Ret.-Ret. and Ret.-Adv. functions



$$E_{\text{bound}} \simeq -2.0001 = -W - \frac{8\lambda^4}{W^3}$$

$-\lambda$
dot ε_d

$$-W/2 \quad W = 2$$

$$E_{\text{res}} \simeq -i0.005 = -i\frac{\lambda^2}{W}$$

$$\lambda = 0.1$$

$$\varepsilon_d = 0$$

$$L_{\text{eff}}^{\text{RA}}(z) = z \implies \pm i0.005,$$

$$L_{\text{eff}}^{\text{RR}}(z) = z \implies \begin{cases} -i0.01 \\ -i0.00382\dots \\ \simeq -i(3 - \sqrt{5})\lambda^2/W \end{cases}$$

Nontrivial eigenvalue!

Summary

- Definition and physics of resonant states
- Time-reversal symmetry breaking
- Nontrivial eigenvalues of the Liouvillian
- A two-level atom in laser?