The Physics of Exceptional Points

Singularities $\Leftrightarrow$ Physics

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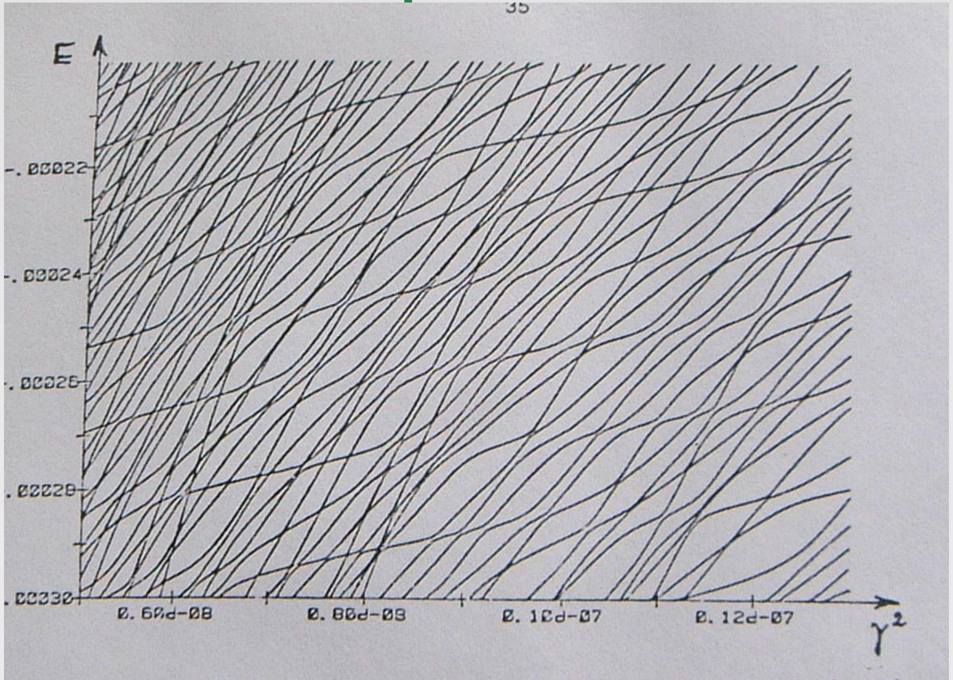
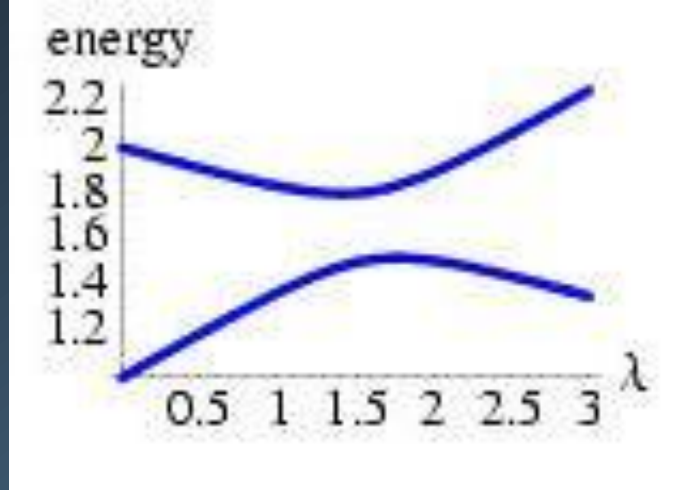
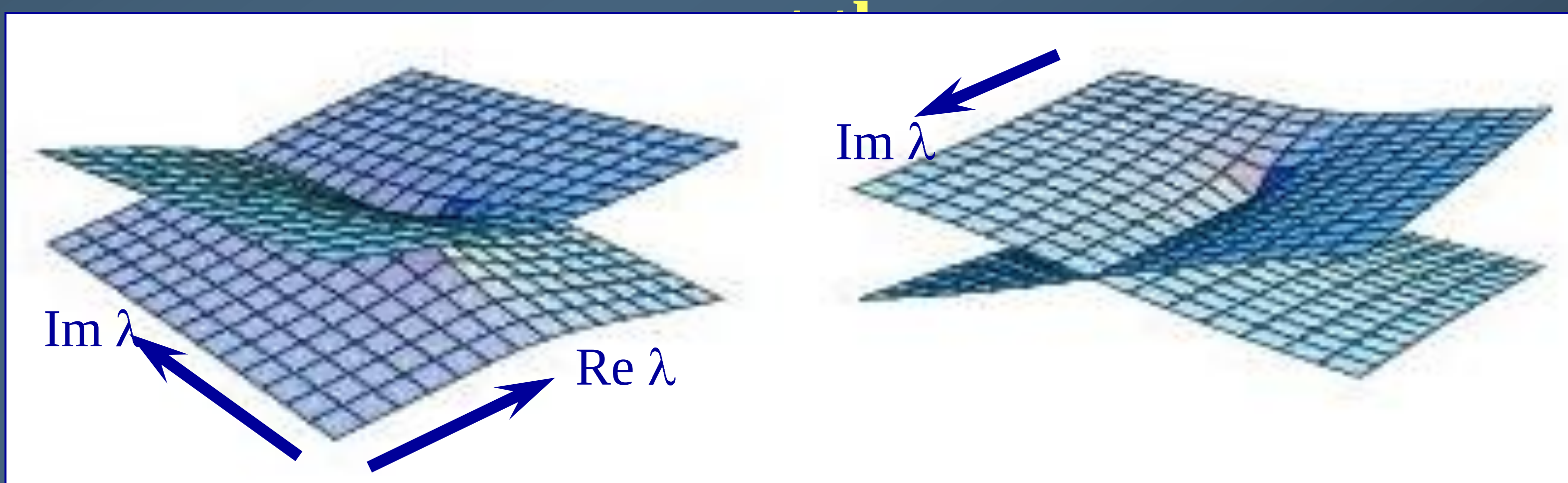
Exceptional Points

The eigenstates of a Hamilton operator, or any matrix depending on a parameter, exhibit level repulsion when sweeping over that parameter





The analytic continuation of the spectrum into the complex λ-plane yields nearby a square root singularity of the energies:
the two repelling levels are analytically connected



The eigenfunction also coalesce,
they have a vanishing norm.

Consequence: ***in vicinity of an EP pattern is determined by the 2-dim sub-problem:***

$$H = \sum_n E_n \frac{|\varphi_n\rangle\langle\varphi^*_n|}{\langle\varphi^*_n|\varphi_n\rangle}$$

when approaching an EP,
where level ***n*** coalesce with level ***n+1***,
the vanishing denominator outweighs all other terms:

you're left with a 2-dim space

For hermitian operators they occur at complex values of such parameters
(rendering the operator non-hermitian at the EP).

For non-hermitian operators (open systems) they also can occur at real parameter values.

At an EP the matrix cannot be diagonalised: only the Jordan-Normal-Form is obtainable by a similarity transformation, i.e.

$$H \approx S \begin{pmatrix} E_{EP} & 1 \\ 0 & E_{EP} \end{pmatrix} S^{-1}$$

- Individual EPs have been established experimentally in the Lab at Darmstadt

- They play a crucial role in quantum phase transitions
  - both in Fermi and in Bose systems -
  and also in the occurrence of chaotic behaviour

- They signal the instability points like in RPA,
  e.g. transition ‘spherical => deformed’ in NuclPhys.
  or the onset of condensation in particular BE-systems,
  or the transition from normal to superconductor,
  or ………..

- They are “in the way” in approximation schemes
  (e.g. intruder states in Nuclear Physics).

- **They signal the symmetry breaking point in $\mathcal{PT}$-symmetric problems**

On this aspect there are many presentations at this conference, also about beautiful experiments in optics, wave guides and others

Here situations should be mentioned with models/arrangements of sources and sinks of, say, atoms where EPs can feature prominently.

- **They can produce dramatic effects in scattering and time behaviour**

Again, during this conference we hear examples, of a general type, and from atomic and molecular physics

- **Special effects can arise in nonlinear problems: e.g the coalescence of three levels (EP3)**

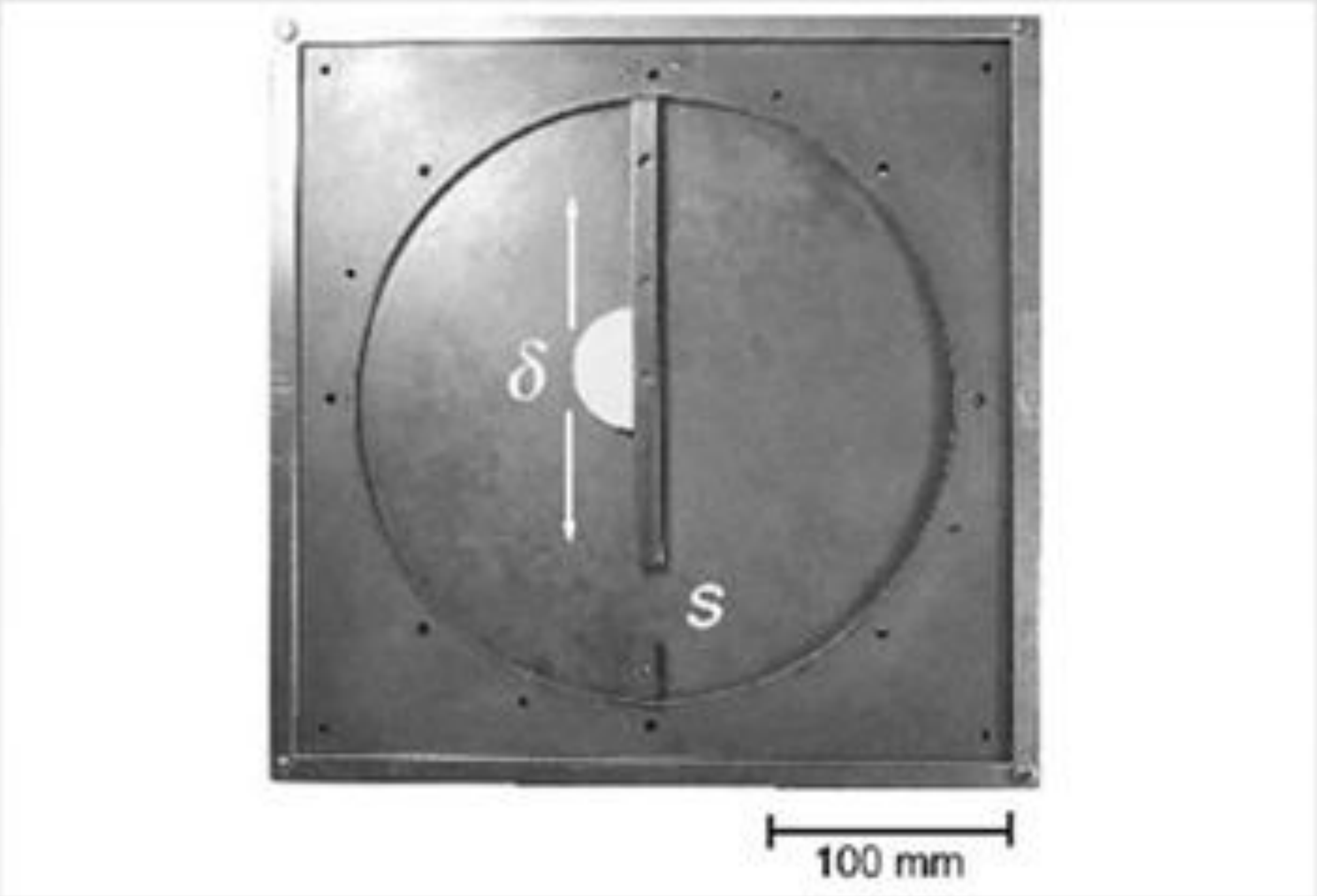
NB:
in linear problems it needs at least five parameters to produce an EP3

The **physical reality** of the Riemann sheet structure has been confirmed experimentally more than ten years ago in **Darmstadt** by **Christian Dembowski** (PhD under **Achim Richter**) where EPs have been encircled using a microwave cavity

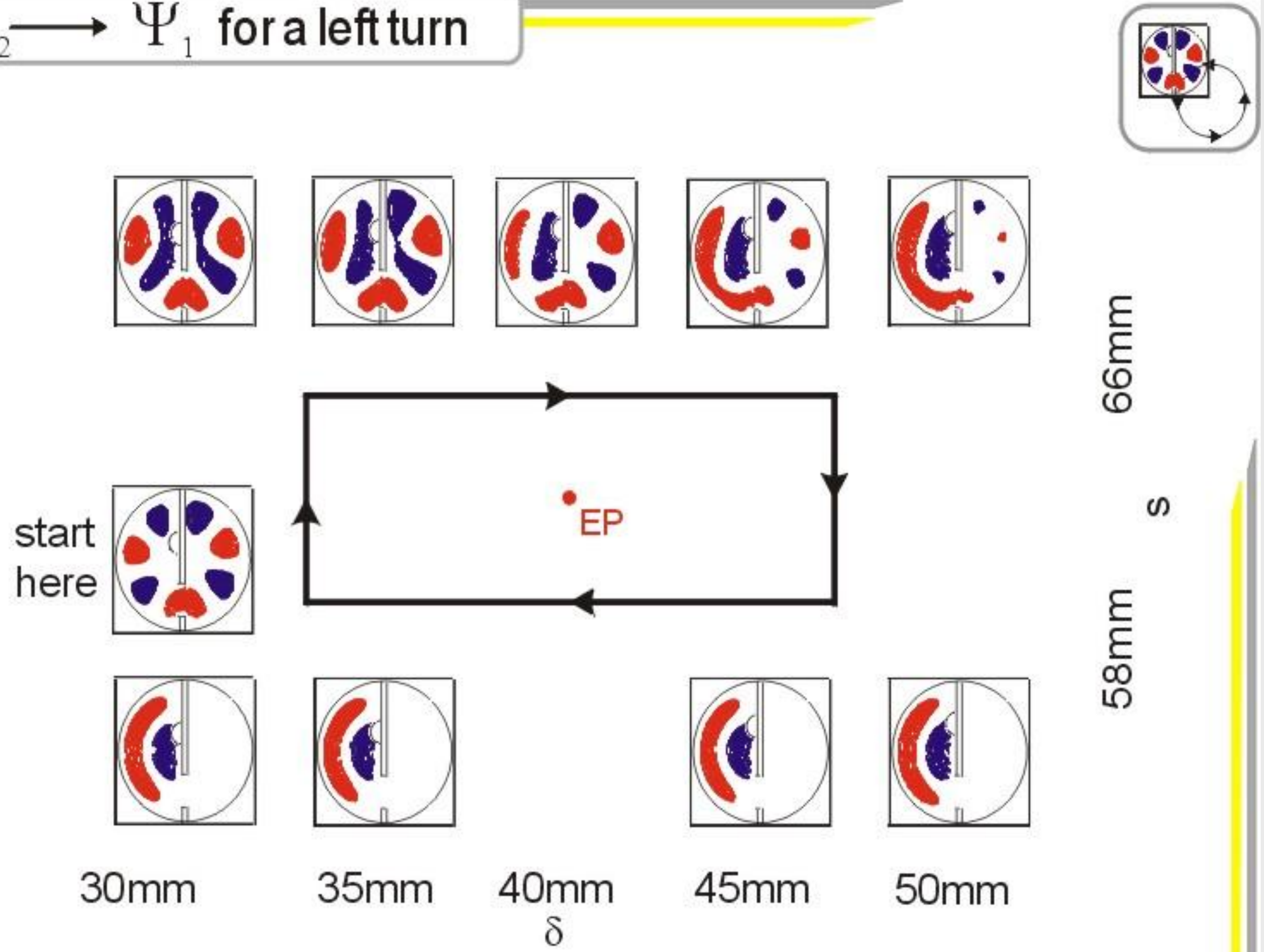
nota bene

an EP can be accessed in the **laboratory only** in a **dissipative (open)** system

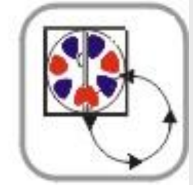
the variable s controls the coupling, and
δ (piece of Teflon) can shift left eigen-mode

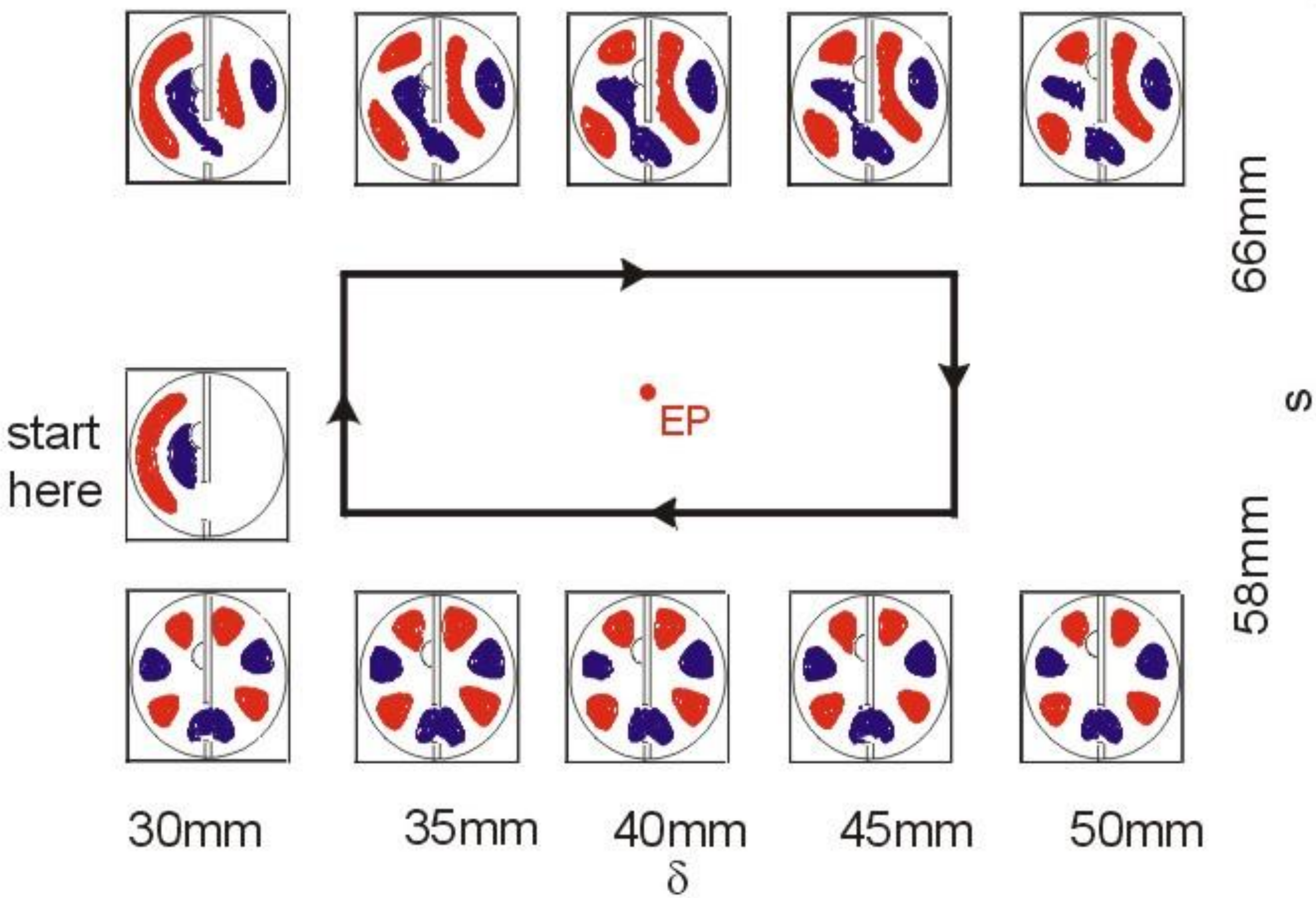


$\Psi_2 \longrightarrow \Psi_1$ for a left turn



$\Psi_1 \longrightarrow -\Psi_2$ for a left turn





We conclude

(1) One loop around EP interchanges energy levels

(2) the state vectors have a 4th root branch point;

when looping one way (say counterclockwise**):**

$$\varphi_1 \to \varphi_2 \to -\varphi_1 \to -\varphi_2 \to \varphi_1$$

and accordingly when looping the other way

(say clockwise):

$$\varphi_1 \to -\varphi_2 \to -\varphi_1 \to \varphi_2 \to \varphi_1$$

some kind of *chirality*

Moreover,
the eigenfunctions **at** the EP also coalesce
and have the form

$$\varphi_1 = \varphi_2 = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Note:
no parameters,
it is cast in stone

which is a truly **chiral** wave function

phase difference of $\pi/2$ **confirmed experimentally in Darmstadt**

Having discussed some aspects of individual Exceptional Points we now turn to their role which they can play collectively.

This applies in particular to many body problems in connection with phase transitions and the onset of chaos.

As an illustration, the Lipkin model:

N Fermions occupying 2 degenerate levels, degeneracy at least *N*-fold.

Interaction lifts or lowers a Fermion pair

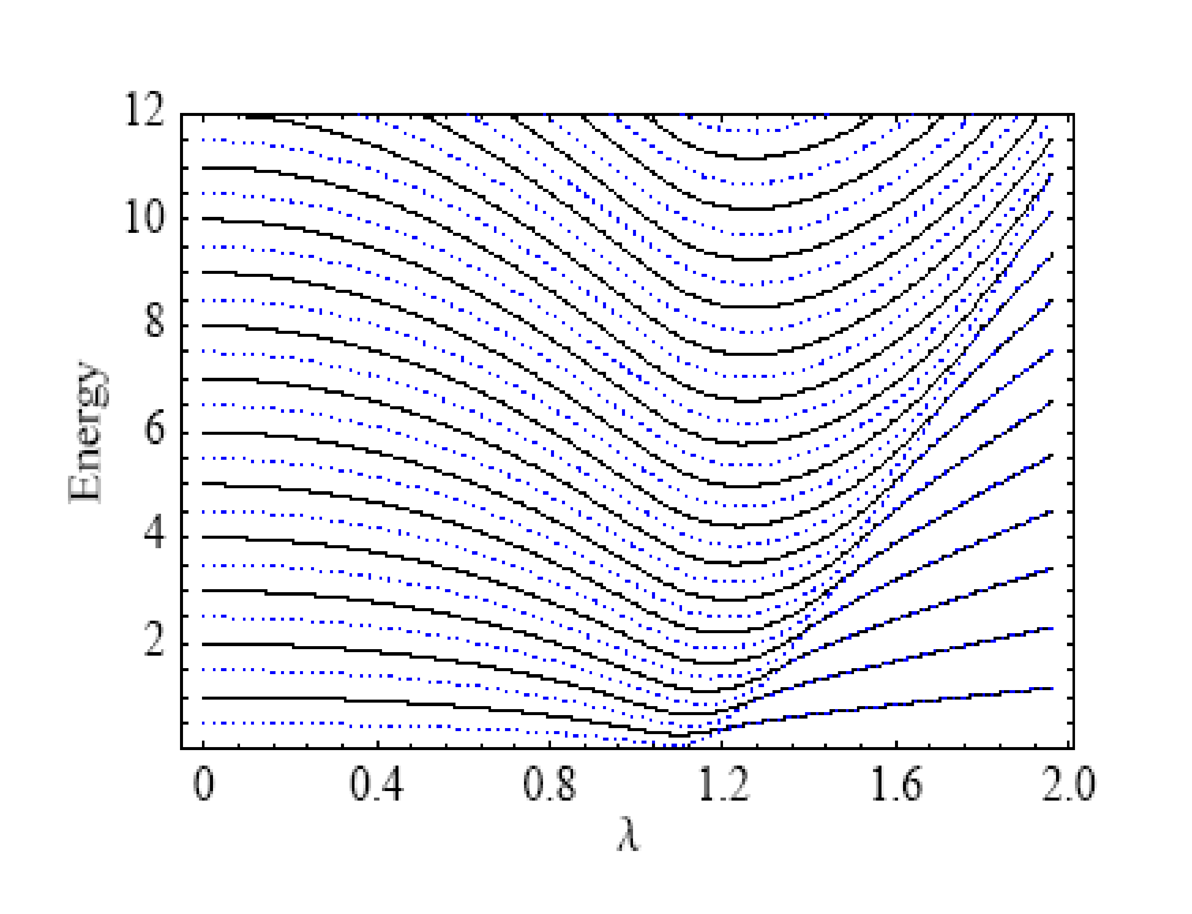
as a consequence:

model is reducible into *even* or *odd N*

$$H = \sum a^{\dagger}_{k,m} a_{k,m} + \lambda \sum a^{\dagger}_{k,m} a^{\dagger}_{k',m'} a_{k',-m'} a_{k,-m}$$

$$H = J_z + \frac{\lambda}{2N}(J_+^2 + J_-^2)$$

model shows phase transition at $\lambda \geq 1$
including ***symmetry breaking*** in that for
$\lambda > 1$ a ‘deformed’ phase occurs
where even and odd N become degenerate



spectrum
with respect
to ground
state

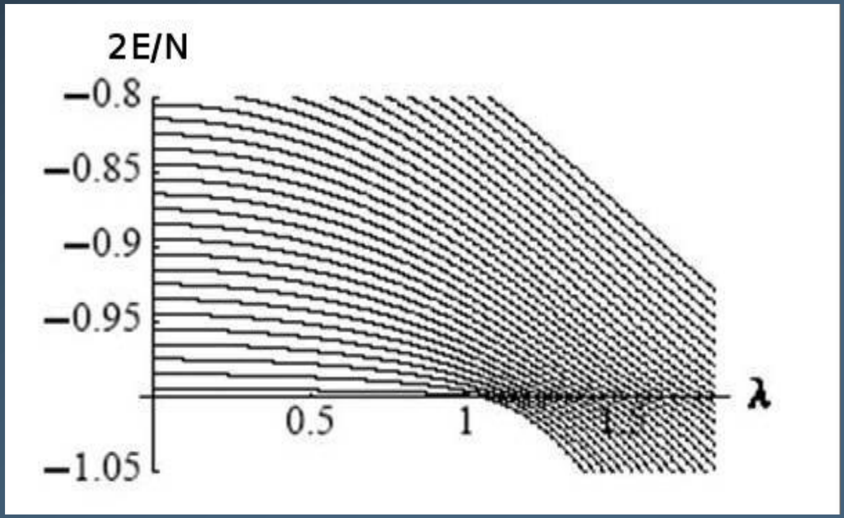
energy gap at the transition point, for large but finite N

$$\Delta E \sim \frac{1}{N^{1/3}} \quad \text{for } \lambda = 1$$

$$\Delta E \sim \frac{\sqrt{\lambda^2 - 1}}{\log N} \quad \text{for } \lambda > 1$$

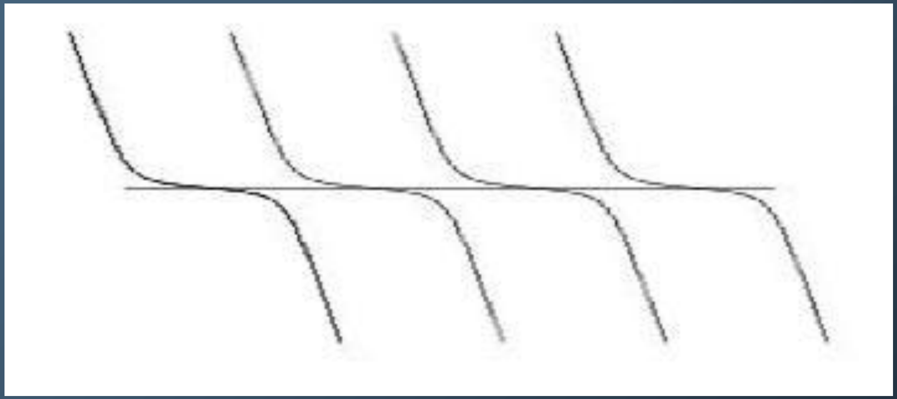
NB: the non-uniform behaviour for $\lambda \to 1$

Spectrum 2E/N as function of λ



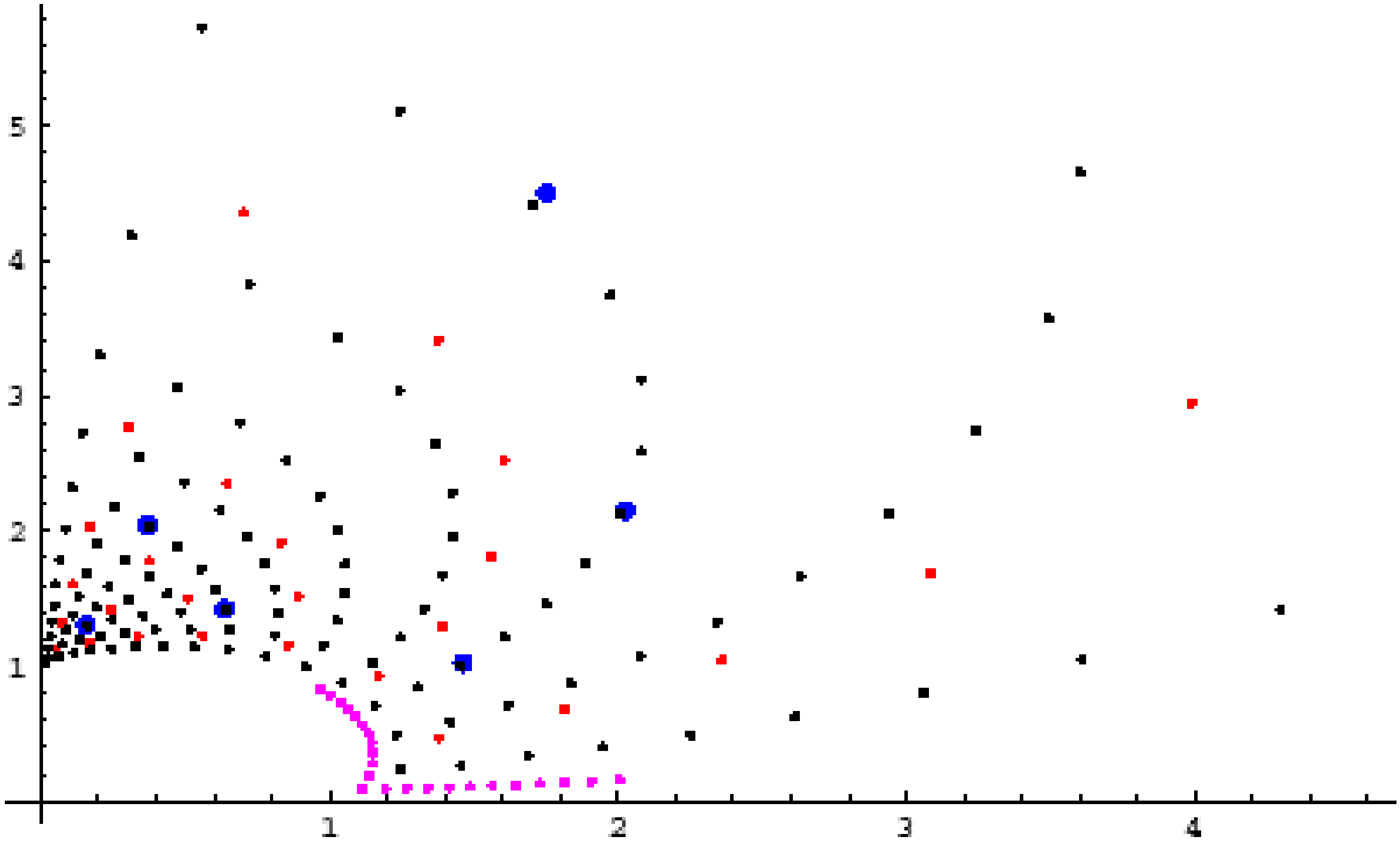
phase transition for all $\lambda>1$ at
$2E/N = -1$

in fact, magnification
along the line
$2E/N = -1$ looks like

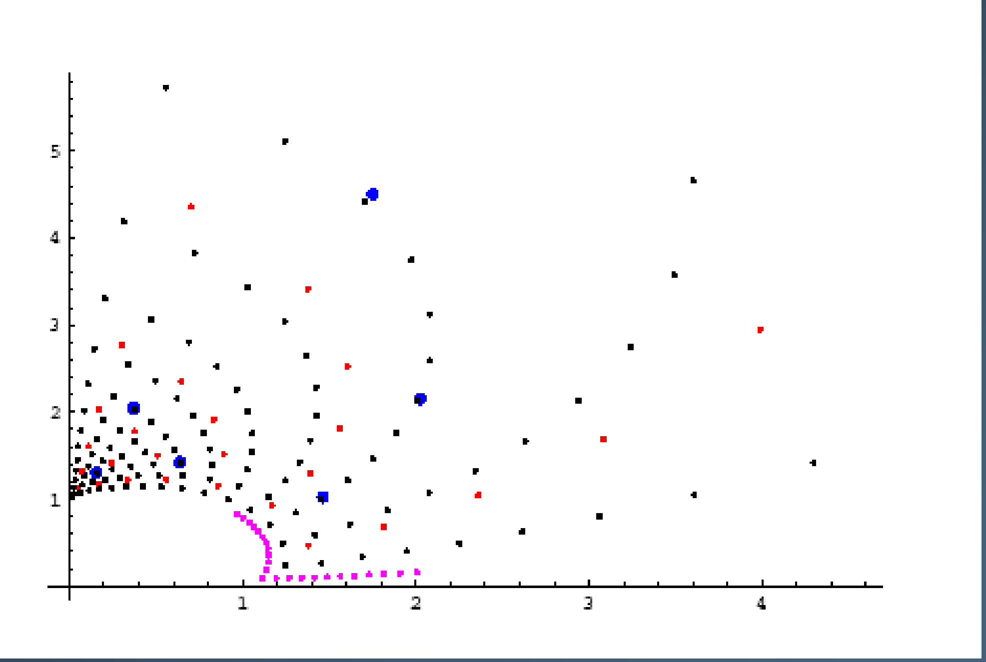


level repulsion – watch EP!

EPs in complex λ - plane for various *N*



N=8 (blue), =16(red), =32(black), =96(pink)



The inner circle

$$|\lambda|<1$$

remains free of singularities

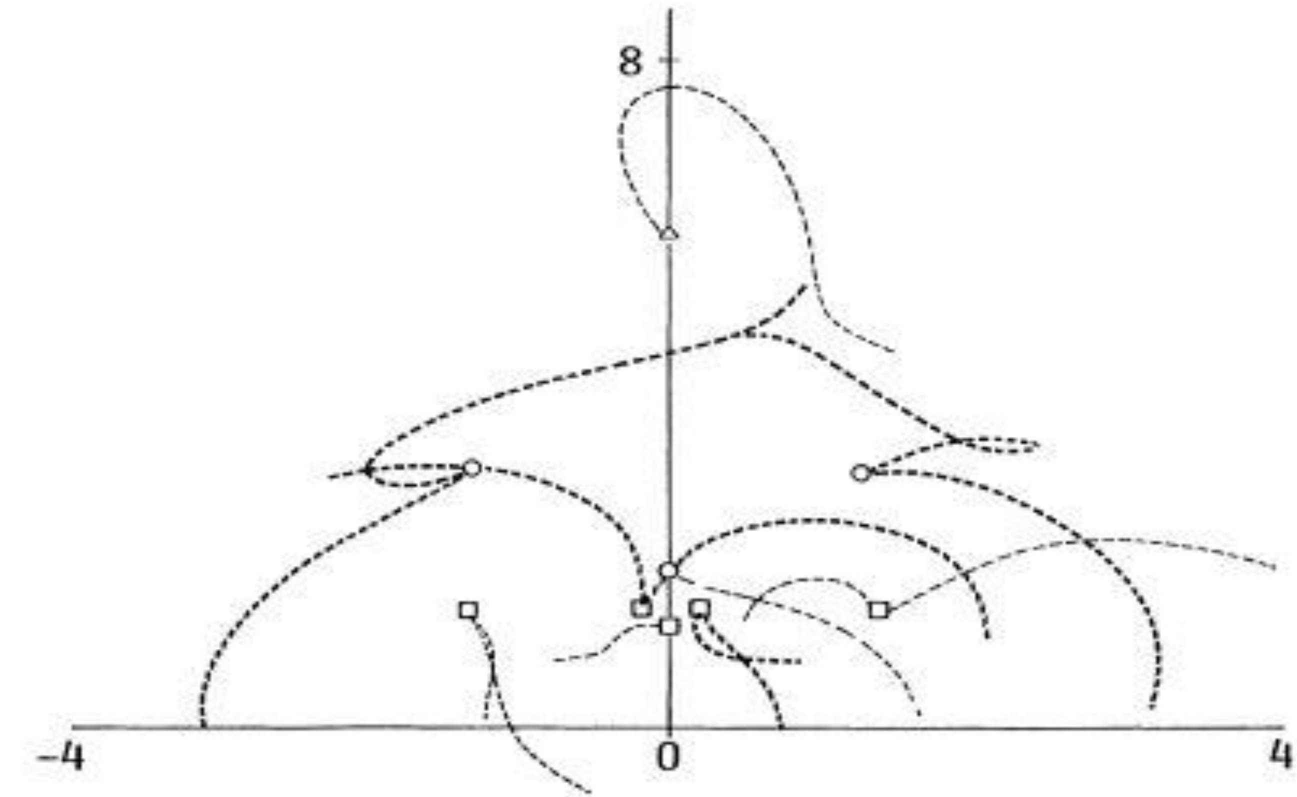
In contrast, for increasing N, the EPs accumulate along the real λ - axis for $\lambda>1$

But: the model yields them nicely ordered. The slightest perturbation whirls them around.

The effect upon the **spectrum:**

Chaotic!

Trajectories of the EPs in the complex λ-plane for N=6 for the perturbation

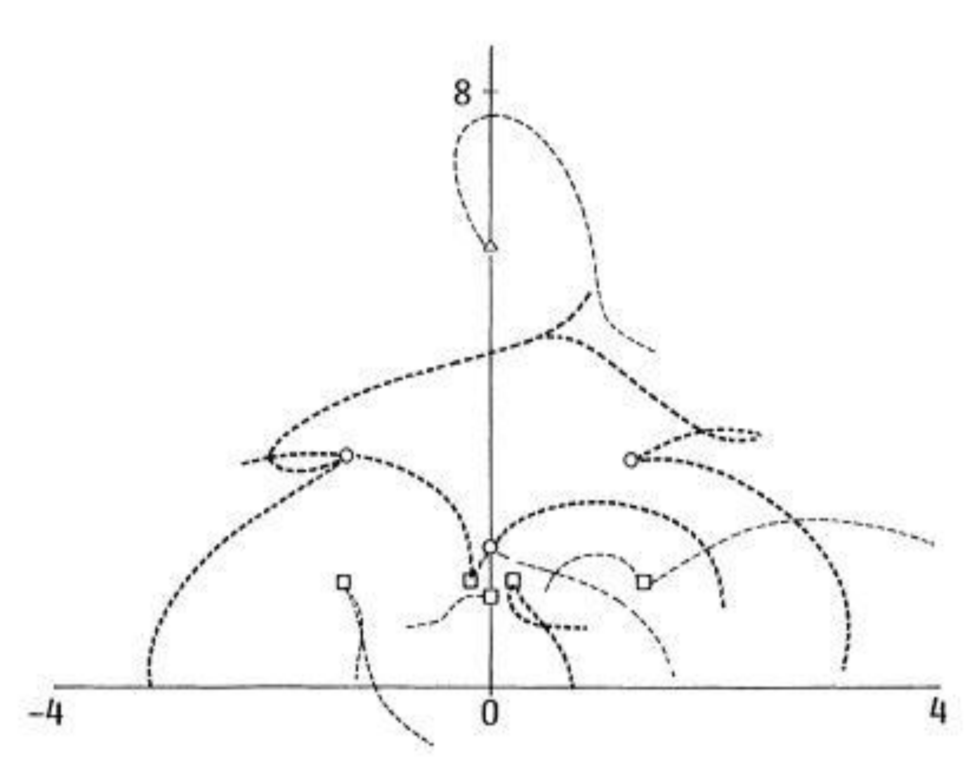


$$J_+^2 + J_-^2 \rightarrow U^\dagger (J_+^2 + J_-^2) U$$

with U a random unitary matrix using random angles from the interval $\{0, \zeta_{max}\}$;

$\zeta_{max} << 1$, i.e. U is close to unity.

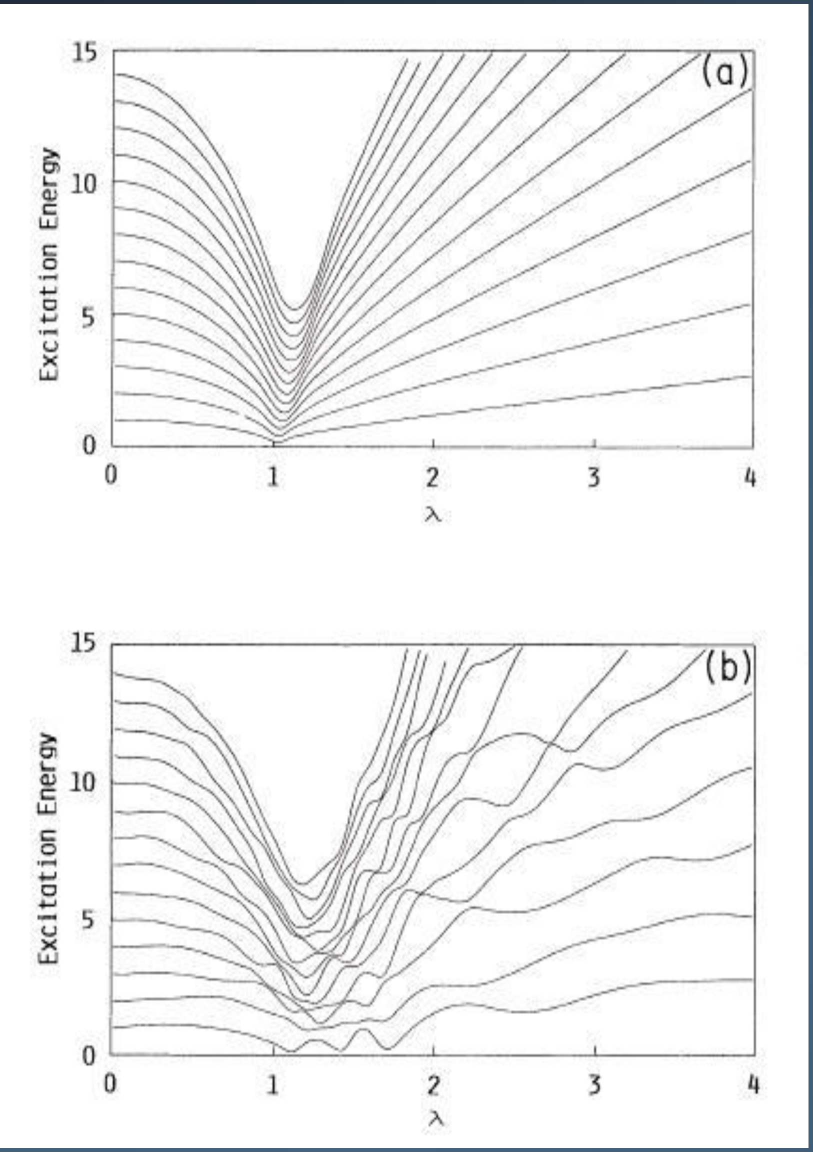
1.The symmetry wrt the imaginary axis is destroyed.



2. Two trajectories emerge from each EP (except on the imaginary axis), i.e.the symmetry around E=0 is also destroyed.

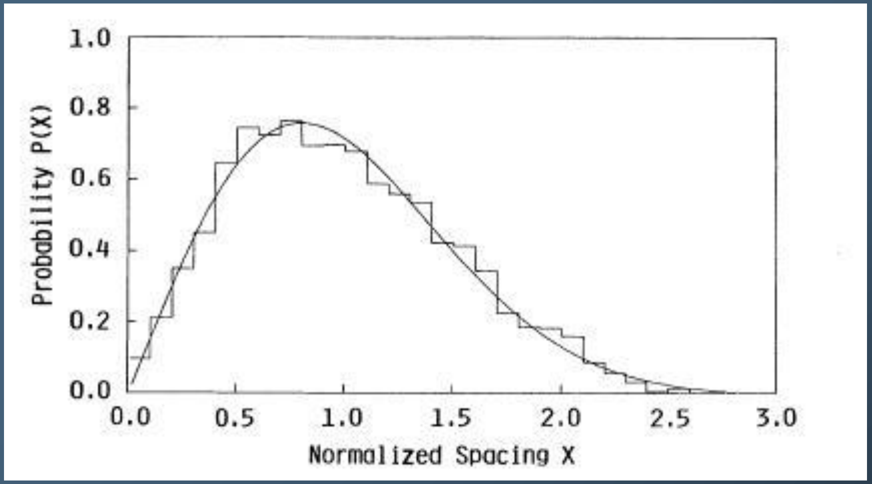
The larger N the smaller ζ_{max} can be chosen to produce typical signatures of chaotic behaviour of spectrum and eigenfcts.

Further increase of ζ_{max} leaves the statistics at λ_{crit} unchanged.



While the level statistics is that of the harmonic oscillator for the unperturbed case, we now obtain the typical Wigner surmise for

$\lambda=\lambda_{crit}$



It is **significant** that this chaotic behaviour does not occur when λ is sufficiently distant from $\lambda_{crit} = 1$; this fact is the more pronounced the larger *N*.

The contribution – linear in the angles of *U* – from the perturbation can be obtained analytically and it turns out that only at λ around λ_{crit} there is an appreciable effect while outside the transitional region the perturbation leaves energy and state vectors virtually unchanged.

1. Due to the absence of **Exceptional Points** the normal phase remains virtually unaffected under small perturbation; so does the deformed region.

2. Transitional regions are the most sensitive against small random perturbation owing to the high density of **Exceptional Points**.

3. Within the transitional region the pattern of the **EP**s looks like

and – for full chaos - the distribution of **EP**s becomes independent of the direction in the λ-plane, is centred around λ_{crit} and given by $1/|\lambda^2|$.

4. The sensitivity and immediate onset of chaos is reminiscent of the classical situation at the crossing point of a separatrix.

5. These findings – high density of EPs- explain the inherent difficulties of many body calculations in transitional regions.

A few considerations about the large N limit:

If the EPs retain their character in the thermodynamic limit

$$N \to \infty$$

the Hamilton-op cannot have

1) an ‘obvious’ self-adjoint limit

‘obvious’: not at all or not unique.

A self-adjoint op cannot have an EP on the real line.

2) the dense population of EPs seems to forbid analytic connectedness;

recall: for finite N, all levels are analytically connected.

A dense set of singularities on a line/curve constitutes a natural boundary of analytic domain

Summary:

The ubiquitous Exceptional Points occur generically in all eigenvalue problems that have some parameter dependence.

They can produce dramatic effects in a great variety of physical problems, individually as well as collectively.

Even in Nuclear Physics – where parameter variation is restricted – they certainly feature in approximation schemes, but probably also directly at the border line of the continuum such as along the drip line.

The End

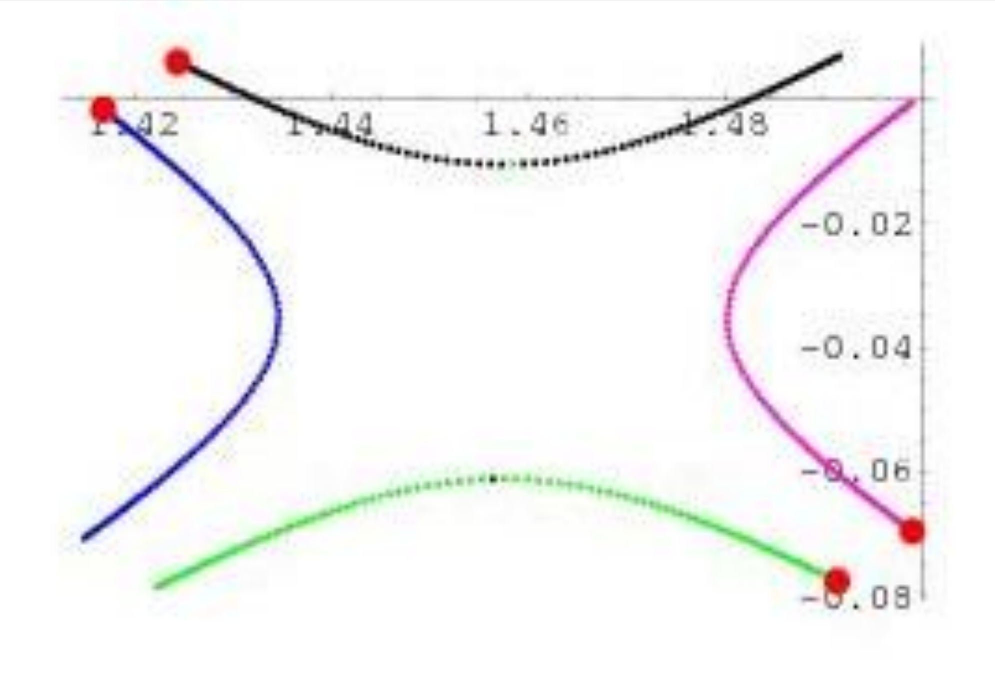
Thank you for your attention

from the Riemann sheet structure we understand:

depending on which side you pass the EP the two

levels repel and the widths cross or

the levels cross and the widths repel



Trajectories in complex energy plane when λ moves from left to right parallel to real axis,

moving between real axis and EP (blue/pink)

moving on the other side of EP (black/green)

Singularities and Zero Energy Bound states

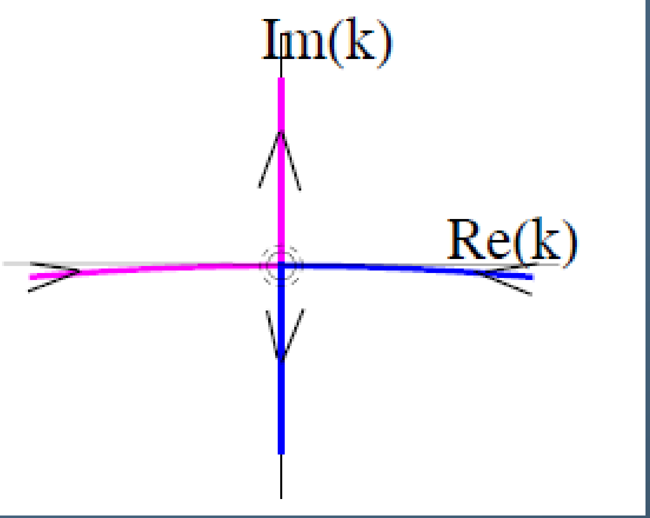
think of shallow nuclei and nuclei on the drip line

A bound state at E=0 ($\ell > 0$) is 'between'
a resonance (potential less attractive)
and a weakly bound state with E>0
(potential more attractive)

Being a 'hybrid', it is a singularity and
thus behaves physically in a conspicuous way

Yet, for a real potential it cannot be an EP
even though it 'looks like' an EP

the eigenvalue at zero is brought about by coalescence under variation of the potential strength



In fact, the leading term of the change in energy shows

- under variation of the potential $(V_0 \triangleq E = 0,\ V_0 \to V_0 + \varepsilon)$
  - the typical square root behaviour (recall $\ell > 0$)

$$k_{(1,2)} = c_{(1,2)}\sqrt{\varepsilon} + O(\varepsilon)$$

yet it is NOT an EP because

In contrast to a genuine EP

- the wave function remains normalisable
  (yet there is only one: no degeneracy!)

- the scattering (Green's) function has no pole at all,
  even though the E=0 state is a bound state

- reason: the spectroscopic factors *vanish*,
  'instead' of going to infinity,
  causing 'usually' a double pole in G

After all:

we deal here with a genuine self adjoint problem
which does not admit an EP

Yet there are nice observable effects:

The cross section for

E=0 (zero energy b.s.)

  $\varepsilon=0$

E<0 (genuine b.s.)

  $\varepsilon>0$

E complex (resonance)

  $\varepsilon<0$

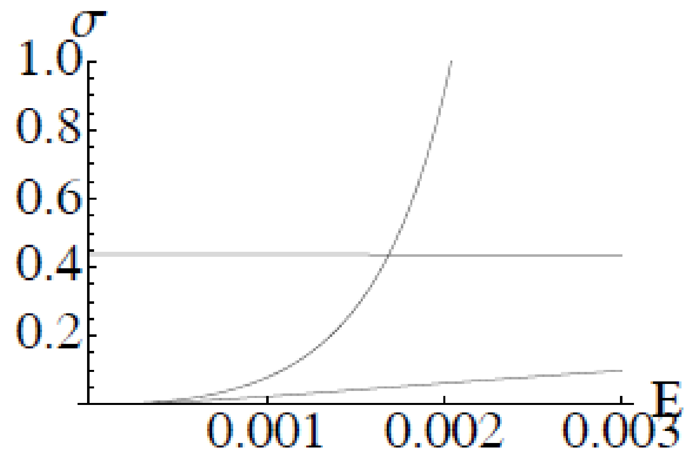


Fig. 2. Cross sections versus energy (in arbitrary units) for $\epsilon = 0$ (straight line), $\epsilon < 0$ (sharp rising curve) and $\epsilon > 0$ (lower curve).

recall $V_0 \to V_0 + \varepsilon$

furthermore recall:

a zero energy bound state wave function falls off only with a power (it is loosely bound),

while – for $\ell > 0$ – it is concentrated on the surface, the more so the larger ℓ

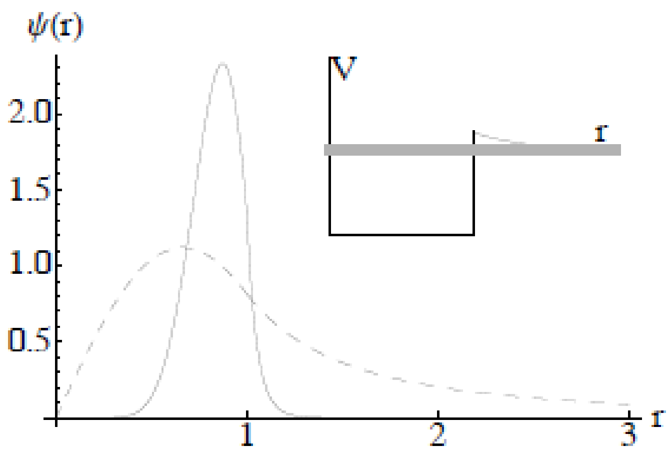


Fig. 3. Normalised zero energy bound state wave functions for $l = 1$ (dashed) and $l = 9$ (solid) for potentials of different depths; here the width of the potential has been chosen unity implying about a factor three for the quotient of the two potential depths. The inset illustrates schematically the effective potential.

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