Pseudo-Hermitian Quantum Systems Defined by an Unbounded Metric Operator

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Contents:

- Background from functional analysis/operator theory
- Pseudo-Hermitian QM with bounded metric operators
- Unbounded metric operators & domain problems
- Pseudo-Hermitian QM with unbounded metric operators
- Application to a simple example

Warning: Use covariant (basis-independent) description of operators:

- Operator $\neq$ Matrix
- H^{T*} is meaningless for an operator H.
- Define $H^\dagger$ covariantly.

Inner-Product & Hilbert Spaces:

- $(V, \langle\cdot|\cdot\rangle)$: An inner product space
- $\| v \| = \sqrt{\langle v|v\rangle}$: Norm of v
- $\{v_n\}$: Convergent sequence if $\exists v \in V$, $\lim\limits_{n\to\infty} \| v_n - v \| = 0$.
- $\{v_n\}$: Cauchy sequence if $\lim\limits_{m,n\to\infty} \| v_m - v_n \| = 0$.
- $\mathscr{H} = (V, \langle\cdot|\cdot\rangle)$: Hilbert space if every Cauchy seq. converges.

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- $\mathscr{H} = (V, \langle\cdot|\cdot\rangle)$: Hilbert space if every Cauchy seq. converges.

- $A \subseteq V$ is a dense subset, if $\forall v \in V$, $\exists$ a sequence $\{v_n\}$ in A such that $v_n \to v$.
- $C \subseteq V$ is a closed subset, if the limit of every convergent sequence in C belongs to C.
- Every inner product space V can be uniquely extended to a Hilbert space $\mathscr{H}$ such that V is dense in $\mathscr{H}$. $\mathscr{H}$ is called the Cauchy completion of V.

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- $L : \mathscr{H} \to \mathscr{H}$ be a linear operator with a dense domain $\mathscr{D}$ &

$$\mathscr{D}' := \left\{\psi \in \mathscr{H} | \exists \xi \in \mathscr{H}, \forall \phi \in \mathscr{D}, \langle \psi | L\phi \rangle = \langle \xi | \phi \rangle \right\}.$$

The adjoint of L is the function $L^\dagger : \mathscr{H} \to \mathscr{H}$ with domain $\mathscr{D}'$ that satisfies: $\forall \psi \in \mathscr{D}'$ & $\forall \phi \in \mathscr{D}$, $\langle \psi | L\phi \rangle = \langle L^\dagger \psi | \phi \rangle$.

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- L is symmetric operator if $\forall \psi, \phi \in \mathscr{D}$, $\langle \phi | L\psi \rangle = \langle L\phi | \psi \rangle$.

- L is self-adjoint or Hermitian, if it is symmetric and $\mathscr{D}' = \mathscr{D}$, i.e., $L^\dagger = L$.

- $L : \mathscr{H}_1 \to \mathscr{H}_2$ is bounded, if

$$\exists c \in \mathbb{R}^+, \ \forall \psi \in \mathscr{D}, \ \langle L\psi | L\psi \rangle_2 \le c \langle \psi | \psi \rangle_1 .$$

- L is continuous, if for every sequence $\{\xi_n\}$ in $\mathscr{D}$ and $\xi \in \mathscr{D}$, $\xi_n \to \xi$ implies $L\xi_n \to L\xi$.

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- Hermiticity & $\mathscr{D} = \mathscr{H}$ imply boundedness (Hellinger-Toeplitz).

- For an unbounded Hermitian operator, $\mathscr{D} \subsetneq \mathscr{H}$.

- $U : \mathscr{H}_1 \to \mathscr{H}_2$ is an isometry, if $\mathrm{Dom}(U) = \mathscr{H}_1$ and

$$\forall \psi_1, \phi_1, \mathscr{H}_1, \quad \langle \phi_1 | \psi_1 \rangle_1 = \langle U\phi_1 | U\psi_1 \rangle_2.$$

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- $U : \mathscr{H}_1 \to \mathscr{H}_2$ is a unitary operator, if it is an onto isometry.

- $A : \mathscr{H} \to \mathscr{H}$ is an automorphism, if it is a one-to-one and onto linear operator $A : \mathscr{H} \to \mathscr{H}$ with domain $\mathscr{H}$ (a linear bijection).

- $L : \mathscr{H} \to \mathscr{H}$ is a positive operator, if it is a Hermitian operator such that $\forall \psi \in \mathscr{D}$, $\langle \psi | L\psi \rangle \geq 0$. It is positive-definite, if $\langle \psi | L\psi \rangle = 0$ only for $\psi = 0$.

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- bounded metric operator := positive-definite automorphism

Hellinger-Toeplitz $\Rightarrow$ Hermitian automorphisms are bounded.

- $H : \mathscr{H} \to \mathscr{H}$ is pseudo-Hermitian if there is a Hermitian automorphism $\eta : \mathscr{H} \to \mathscr{H}$ such that $H^\dagger = \eta\, H\, \eta^{-1}$ or $\eta H = H^\dagger \eta$.

- For diagonalizable linear operators with a discrete spectrum,

Pseudo-Hermiticity $\Leftrightarrow$ Antilinear Symmetries

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- $H : \mathscr{H} \to \mathscr{H}$ is quasi-Hermitian if there is a positive-definite automorphism $\eta_+ : \mathscr{H} \to \mathscr{H}$ such that $H^\dagger = \eta_+\, H\, \eta_+^{-1}$ or $\eta_+ H = H^\dagger \eta_+$.

- For a linear operator with a discrete spectrum,

  Quasi-Hermiticity $\Leftrightarrow$ Diagonalizablity + Reality of Spectrum

Pseudo-Hermiticity versus PT-Symmetry, JMP **43** (2002) 205, math-ph/0107001.

Pseudo-Hermiticity versus PT-Symmetry II, JMP **43** (2002) 2814, math-ph/0110016.

Pseudo-Hermiticity versus PT-Symmetry III, JMP **43** (2002) 3944, math-ph/0203005.

Consequences:

- Role of **antilinear symmetries** such as $\mathcal{PT}$-symmetry
- Construction of metric operators (**non-uniqueness**)

Applications:

- **RQM**: Probabilistic Interpretation of KG fields (2003) & Proca fields (2009)
- Hilbert-space problem in **quantum cosmology** (2003-2004)
- **Electrodynamics**: Permeability tensor as a metric operator (2008-2010)
- Physics of **Spectral Singularities** (2009): Threshold Lasing & Antilasing

Pseudo-Hermitian QM: Given a quasi-Hermitian operator $H : \mathscr{H} \to \mathscr{H}$ and a corresponding (bounded) metric operator η_+, one can redefine the inner-product of the Hilbert space, $$\langle\phi|\psi\rangle \to \langle\psi,\psi\rangle_{\eta_+} := \langle\phi|\eta_+\psi\rangle,$$ such that $H : \mathscr{H}_{\eta_+} \to \mathscr{H}_{\eta_+}$ is Hermitian.

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- $\mathscr{H}$ and $\mathscr{H}_{\eta_+}$ are the same as sets and topological vector spaces, but different as inner-product spaces.

- $\rho := \sqrt{\eta_+} : \mathscr{H}_{\eta_+} \to \mathscr{H}$ is a unitary operator.
- $h := \rho\, H\, \rho^{-1} : \mathscr{H} \to \mathscr{H}$ is Hermitian.

$\rho := \sqrt{\eta_+}$

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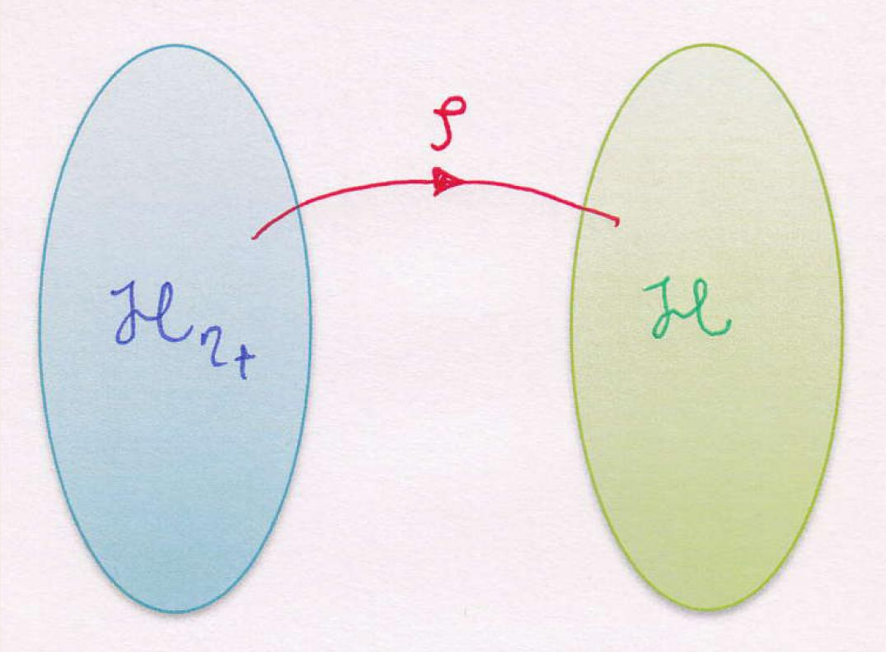

$H : \mathscr{H}_{\eta_+} \to \mathscr{H}_{\eta_+}$

$h : \mathscr{H} \to \mathscr{H}$

$(\mathscr{H}_{\eta_+}, H)$ and $(\mathscr{H}, h)$ are unitary-equivalent.

They describe the same physical system.

Pseudo-Hermitian Representation of the system:

- Physical Hilbert space: $\mathscr{H}_{\eta_+}$
- Observables: Hermitian operators $O:\mathscr{H}_{\eta_+}\to\mathscr{H}_{\eta_+}$
- Hamiltonian: $H:\mathscr{H}_{\eta_+}\to\mathscr{H}_{\eta_+}$

Hermitian Representation of the system:

- Physical Hilbert space: $\mathscr{H}$
- Observables: Hermitian operators $o:\mathscr{H}\to\mathscr{H}$
- Hamiltonian: $h:=\rho\,H\,\rho^{-1}:\mathscr{H}\to\mathscr{H}$ $\quad(\rho:=\sqrt{\eta_+})$

The central ingredient of pseudo-Hermitian QM is the metric operator. Its choice is restricted by the Hamiltonian via the pseudo-Hermiticity relation $H^\dagger = \eta_+ H \eta_+^{-1}$, but it is not unique.

Different choices of η_+ determine different quantum systems with the same Hamiltonian H but different Hilbert space $\mathscr{H}_{\eta_+}$.

δ-Function Potential with Complex Coupling

$$H = \frac{p^2}{2m} + \zeta\,\delta(x), \qquad \zeta \in \mathbb{C}, \qquad \Re(\zeta) > 0$$

H is not $\mathcal{PT}$-symmetric.

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H is not $\mathcal{PT}$-symmetric.

We can obtain a perturbative expansion for a metric operator η_+ that gives:

$$h = \frac{p^2}{2m} + \Re(\zeta)\,\delta(x) + \Im(\zeta)^2\; h_2 + \mathcal{O}(\Im(\zeta)^3)$$

h is a nonlocal operator.

JPA 39 (2006) 13495, quant-ph/0606198

$\mathcal{PT}$-Symmetric Anharmonic Oscillator:

[Bender & Boettcher, PRL 80 (1998) 5243]

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$$H = \frac{p^2}{2m} + \frac{\mu^2}{2}x^2 + \mathrm{i}\epsilon x^3$$

Using perturbation theory we find a particular η_+ that gives:

$$h = \frac{p^2}{2m} + \frac{1}{2}\mu^2 x^2 + \frac{3}{2\mu^4}\left(\frac{1}{m}\{x^2, p^2\} + \mu^2 x^4 + \frac{2\hbar^2}{3m}\right)\epsilon^2 + \frac{2}{\mu^{12}}\left(\frac{p^6}{m^3} - \frac{9\mu^2}{m^2}\{x^2, p^4\} - \frac{51\mu^4}{8m}\{x^4, p^2\} - \frac{7\mu^6}{4}x^6 - \frac{81\hbar^2\mu^2}{2m^2}p^2 - \frac{69\hbar^2\mu^4}{2m}x^2\right)\epsilon^4 + \mathcal{O}(\epsilon^6)$$

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[JPA 38 (2005) 6557]

Problem: The metric operators we could construct for this and almost all other quasi-Hermitian Hamiltonian operators that act in an ∞-dim. Hilbert space are unbounded operators!

$\Rightarrow$ $\mathrm{dom}(\eta_+) \subsetneq \mathscr{H}$

$\Rightarrow$ $\exists\psi \in \mathscr{H},\ \eta_+\psi$ does not exist.

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Impractical solutions:

- Kretschmer & Szymanowski, PLA **325**, 112 (2004).
- Mostafazadeh & Batal, JPA **37**, 11645 (2004).
- Mostafazadeh, IJGMMP **7**, 1191 (2010); arXiv:0810.5643.

Construction of $\mathscr{H}_{\eta_+}$ & h ?

- Unbounded metric operator := unbounded positive-definite operator

- H is η_+-pseudo-Hermitian operator, if $H^\dagger\eta_+ = \eta_+ H$.

- Both H and η_+ act in $\mathscr{H}$ and have dense domains.

- $\eta_+ > 0 \Rightarrow \forall\psi \in \mathrm{dom}(\eta_+),\ \psi \neq 0 \Rightarrow \langle\psi|\eta_+\psi\rangle \in \mathbb{R}^+$.

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- $\rho := \sqrt{\eta_+} : \mathscr{H} \to \mathscr{H}$ is also positive-definite.

- $\eta_+ = \rho^2 \Rightarrow \mathrm{dom}(\eta_+) \subseteq \mathrm{dom}(\rho) \subsetneq \mathscr{H}$.

- η_+ and ρ are both invertible.

- η_+ and ρ are both Hermitian $\Rightarrow$

$$\forall\phi,\psi \in \mathrm{dom}(\eta_+),\ \langle\phi|\eta_+\psi\rangle = \langle\phi|\rho^2\psi\rangle = \langle\rho\phi|\rho\psi\rangle$$

Assumptions:

1) H has a real and discrete spectrum with eigenvectors ψ_n whose span

$$\mathcal{S} := \left\{ \sum_{n=0}^{N} c_n \psi_n \;\middle|\; N \in \mathbb{N},\ c_n \in \mathbb{C} \right\} \subsetneq \mathscr{H}$$

is an ∞-dim. subspace of $\mathscr{H}$.

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2) $\exists \eta_+ > 0,\ H^\dagger \eta_+ = \eta_+ H$.

3) $\psi_n \in \mathrm{dom}(\eta_+) \Rightarrow \mathcal{S} \subseteq \mathrm{dom}(\eta_+) \subseteq \mathrm{dom}(\rho)$

$$\rho(\mathcal{S}) := \left\{ \sum_{n=0}^{N} c_n \rho \psi_n \;\middle|\; N \in \mathbb{N},\ c_n \in \mathbb{C} \right\}$$

Because ρ is one-to-one, $\rho(\mathcal{S})$ is an ∞-dim. subspace of $\mathscr{H}$.

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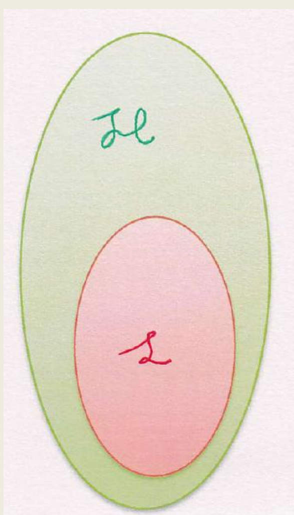
Aim: To construct analogs of $\mathscr{H}_{\eta_+}$ and h, and use them to define a quantum system.

Construction of $\mathscr{H}_{\eta_+}$ & h:

Step 1) Let $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ be the inner-product on $\mathcal{S}$ defined by

$$\forall\phi,\psi\in\mathcal{S},\ \ \langle\!\langle\phi,\psi\rangle\!\rangle := \langle\phi|\eta_+\psi\rangle = \langle\rho\phi|\rho\psi\rangle.$$

Then $\mathcal{S}$ is an inner-product space.



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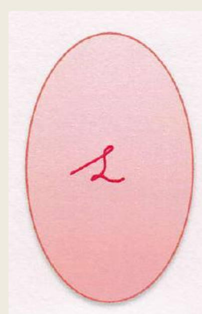

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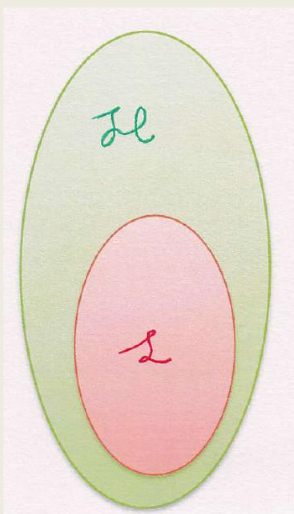
Step 2) $\mathscr{H}_{\eta_+}$:= Cauchy completion of $\mathcal{S}$



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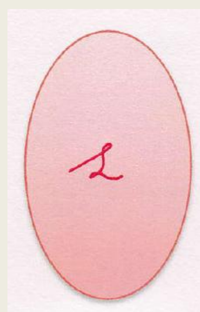

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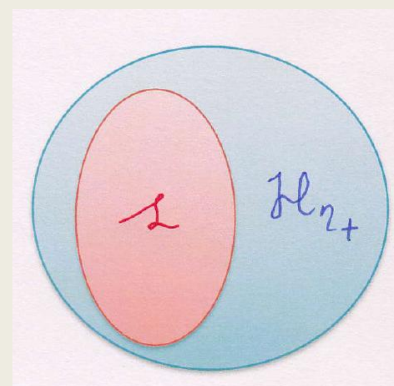

Cauchy-Completed $\mathcal{S}$

Step 3) Restrict $H: \mathscr{H} \to \mathscr{H}$ to $\mathcal{S}$. This defines an operator with domain $\mathcal{S}$ which is dense in $\mathscr{H}_{\eta_+}$. We also denote it by H. Because $H^\dagger \eta_+ = \eta_+ H$, $\forall \phi, \psi \in \mathcal{S}$,

$$\langle\!\langle \phi, H\psi \rangle\!\rangle = \langle \phi | \eta_+ H\psi \rangle = \langle \phi | H^\dagger \eta_+ \psi \rangle = \langle H\phi | \eta_+ \psi \rangle = \langle\!\langle H\phi, \psi \rangle\!\rangle.$$

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We can construct an orthonormal basis $\{\psi_n\}$ of $\mathscr{H}_{\eta_+}$ using eigenvectors of H.

$H : \mathscr{H}_{\eta_+} \to \mathscr{H}_{\eta_+}$ is symmetric but not Hermitian.

Must find a Hermitian (self-adjoint) extension of H.

Step 4) Recall $H\psi_n = E_n\psi_n$. Let

$$\mathscr{D} := \left\{ \sum_{n=0}^{\infty} a_n\psi_n \;\middle|\; \sum_{n=0}^{\infty} E_n^2|a_n|^2 < \infty \right\},$$

and define $\hat{H} : \mathscr{H}_{\eta_+} \to \mathscr{H}_{\eta_+}$ with domain $\mathscr{D}$ by

$$\hat{H}\left(\sum_{n=0}^{\infty} a_n\psi_n\right) := \sum_{n=0}^{\infty} E_n a_n \psi_n.$$

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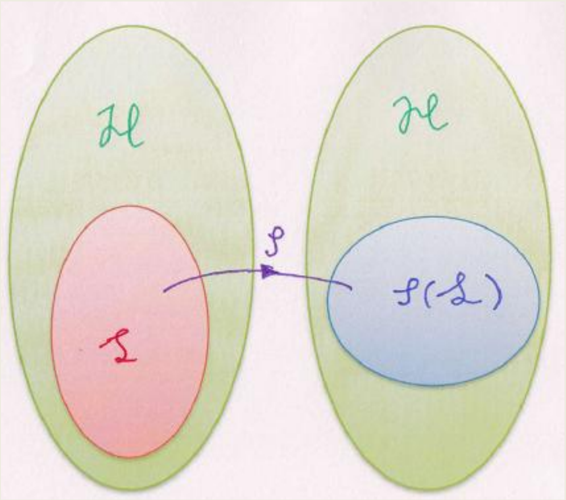
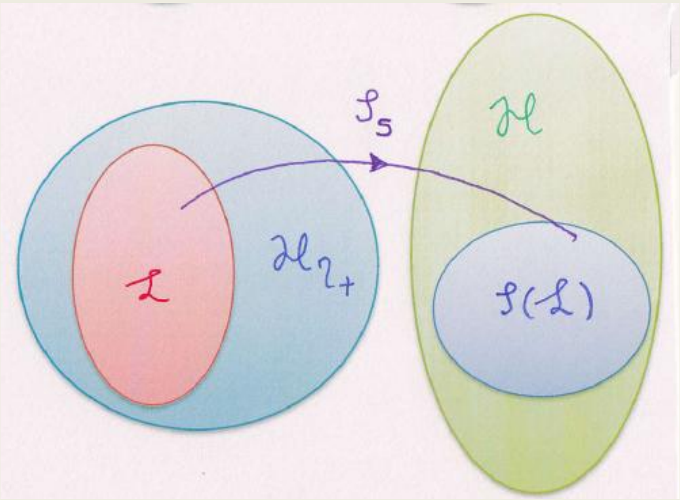
$(\mathscr{H}_{\eta_+}, \hat{H})$ defines a unitary quantum system.

$\mathscr{H}_{\eta_+}$ and $\mathscr{H}$ are different even as sets.

Step 5) Restrict $\rho : \mathscr{H} \to \mathscr{H}$ to $\mathcal{S}$. This gives an operator $\rho_s : \mathscr{H}_{\eta_+} \to \mathscr{H}$ with domain $\mathcal{S}$.

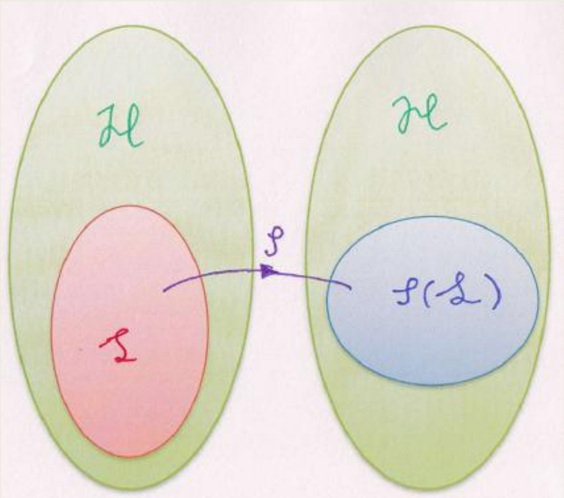


$\rho : \mathscr{H} \to \mathscr{H}$



$\rho_s : \mathscr{H}_{\eta_+} \to \mathscr{H}$

Step 5) Restrict $\rho : \mathscr{H} \to \mathscr{H}$ to $\mathcal{S}$. This gives an operator $\rho_s : \mathscr{H}_{\eta_+} \to \mathscr{H}$ with domain $\mathcal{S}$.



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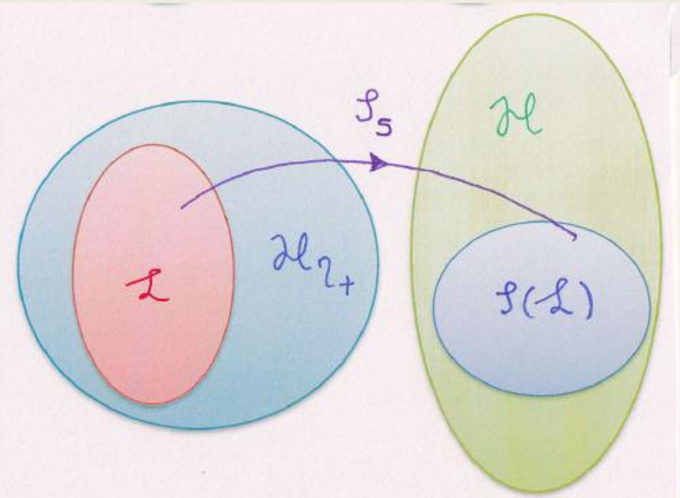


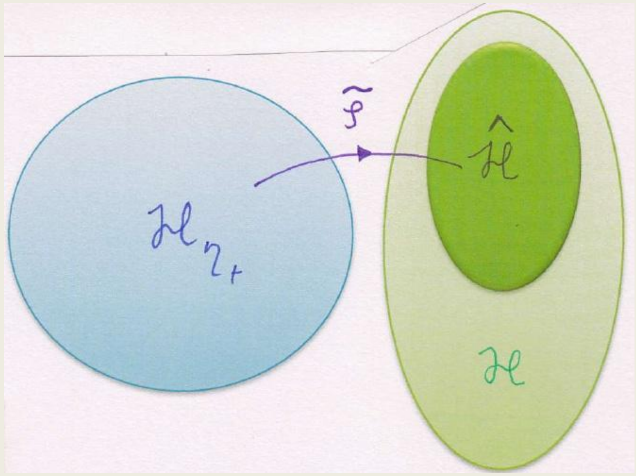
$\rho_s : \mathscr{H}_{\eta_+} \to \mathscr{H}$

$\mathcal{S}$ is dense in $\mathscr{H}_{\eta_+}$ & $\forall \phi, \psi \in \mathcal{S}$, $\langle\!\langle \phi, \psi \rangle\!\rangle = \langle \rho\phi | \rho\psi \rangle = \langle \rho_s\phi | \rho_s\psi \rangle$.
ρ_s is a bounded operator that can be extended to $\mathscr{H}_{\eta_+}$.

This defines an isometry $\tilde{\rho} : \mathscr{H}_{\eta_+} \to \mathscr{H}$.

Can show that range($\tilde{\rho}$) $= \tilde{\rho}(\mathscr{H}_{\eta_+})$ is a closed subspace of $\mathscr{H}$.

$\Rightarrow$ It is a Hilbert space $\widehat{\mathscr{H}}$.

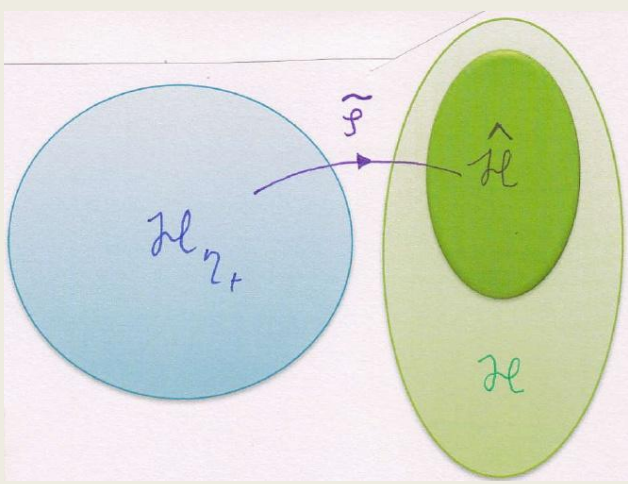


$\tilde{\rho}_s : \mathscr{H}_{\eta_+} \longrightarrow \mathscr{H}$

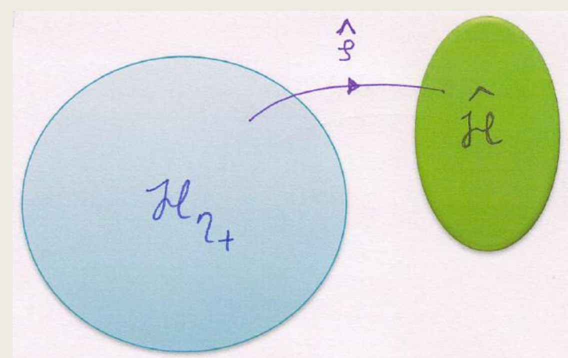
Can show that $\text{range}(\tilde{\rho}) = \tilde{\rho}(\mathscr{H}_{\eta_+})$ is a closed subspace of $\mathscr{H}$.

$\Rightarrow$ It is a Hilbert space $\widehat{\mathscr{H}}$.

Let $\widehat{\rho} : \mathscr{H}_{\eta_+} \to \widehat{\mathscr{H}}$ be defined by $\widehat{\rho}\psi := \tilde{\rho}\psi$ for all $\psi \in \mathscr{H}_{\eta_+}$.



$\tilde{\rho}_s : \mathscr{H}_{\eta_+} \to \mathscr{H}$



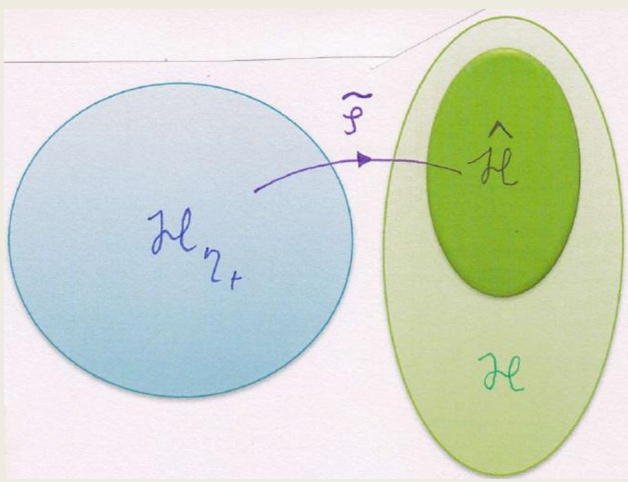
$\widehat{\rho}_s : \mathscr{H}_{\eta_+} \to \widehat{\mathscr{H}}$

Can show that range$(\tilde{\rho}) = \tilde{\rho}(\mathscr{H}_{\eta_+})$ is a closed subspace of $\mathscr{H}$.

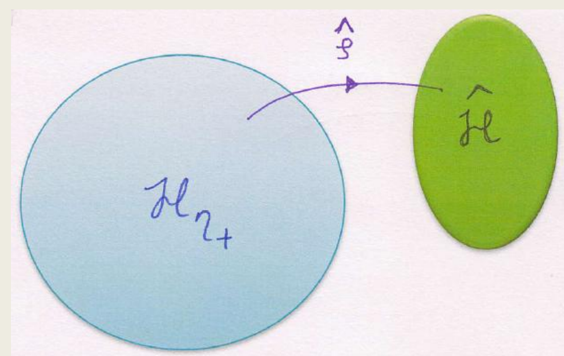
$\Rightarrow$ It is a Hilbert space $\widehat{\mathscr{H}}$.

Let $\hat{\rho} : \mathscr{H}_{\eta_+} \to \widehat{\mathscr{H}}$ be defined by $\hat{\rho}\psi := \tilde{\rho}\psi$ for all $\psi \in \mathscr{H}_{\eta_+}$.

- $\hat{\rho}$ is a unitary operator.
- $h := \hat{\rho}\,\hat{H}\,\hat{\rho}^{-1} : \widehat{\mathscr{H}} \to \widehat{\mathscr{H}}$ is a Hermitian operator.
- $(\widehat{\mathscr{H}}, h)$ defines the same quantum system as $(\mathscr{H}_{\eta_+}, \hat{H})$.



$\tilde{\rho}_s : \mathscr{H}_{\eta_+} \to \mathscr{H}$



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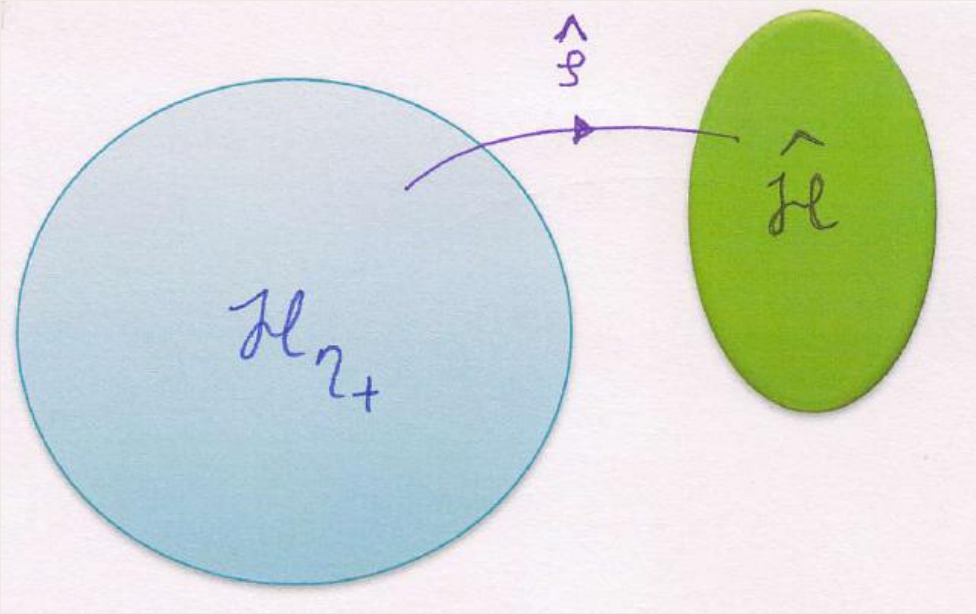

The pseudo-Hermitian quantum system that is defined by H and η_+ can be represented by either of $(\mathscr{H}_{\eta_+}, H)$ or $(\widehat{\mathscr{H}}, h)$.

$$h := \widehat{\rho}\,\widehat{H}\,\widehat{\rho}^{-1}$$

arXiv: 1203.6241 (Phil. Trans. Roy. Soc. A, to appear)

Example:

$$\mathscr{H} := L^2(\mathbb{R}), \qquad H := \frac{1}{2}(p - i\alpha)^2 + V(x)$$

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$$V(x) = \frac{\omega^2 x^2}{2}$$

$$\psi_n(x) := N_n H_n(\sqrt{\omega}\, x) e^{-\frac{\omega x^2}{2} - \alpha x}$$

$$(\eta_+ \psi_n)(x) = N_n H_n(\sqrt{\omega}\, x) e^{-\frac{\omega x^2}{2} + \alpha x}$$

$$\eta_+ \psi_n \in \mathscr{H} \quad \Rightarrow \quad \psi_n \in \mathrm{dom}(\eta_+)$$

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$$\eta_+ \psi_n \in \mathscr{H} \quad \Rightarrow \quad \psi_n \in \mathrm{dom}(\eta_+)$$

$$\langle\!\langle \phi, \psi \rangle\!\rangle := \int_{-\infty}^{\infty} e^{2\alpha x} \phi(x)^* \psi(x) dx$$

$$h = \frac{1}{2}(p^2 + \omega^2 x^2)$$

$(\mathscr{H}_{\eta_+}, H)$ or $(\hat{\mathscr{H}}, h)$ both define a simple harmonic oscillator.

Conclusion: The problem of dealing with unbounded metric operators has been a subject of on-going discussion till the inception of pseudo-Hermitian QM in 2003. We have given a complete resolution of this problem.

Applications of the construction we developed for this purpose awaits further study.

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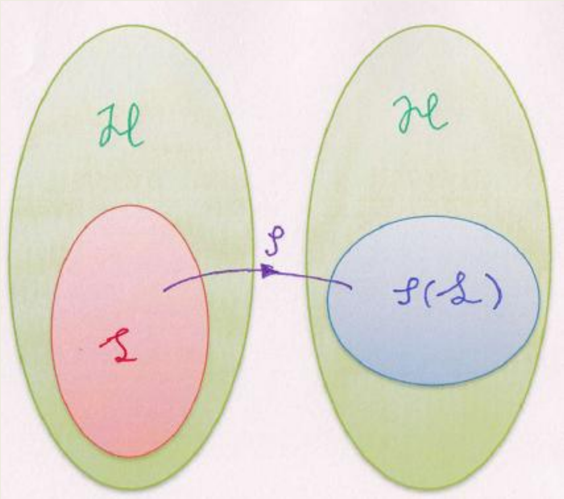
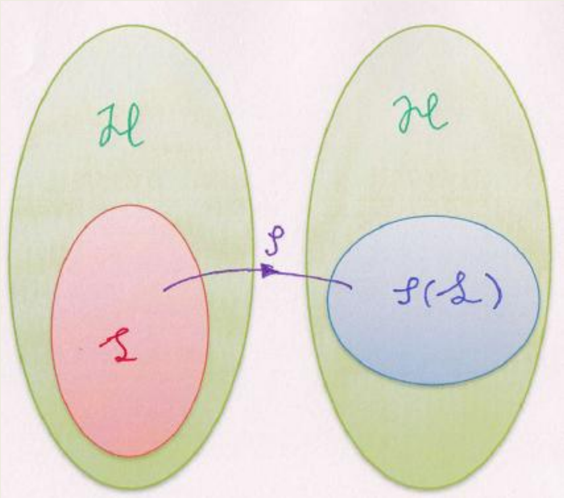
Thank you for your attention.



$$\rho : \mathscr{H} \rightarrow \mathscr{H}$$



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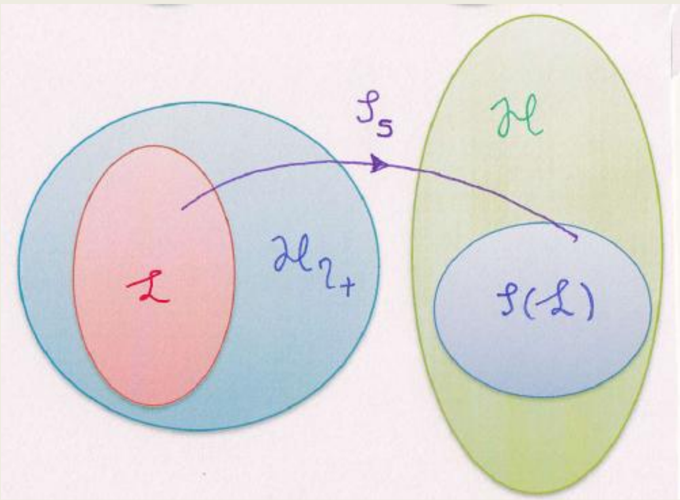
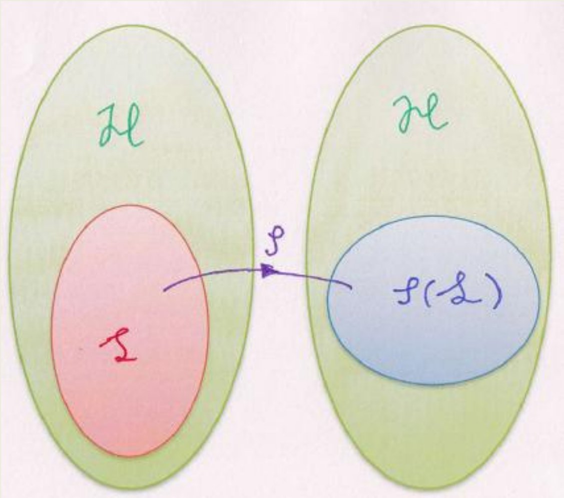


$$\rho_s : \mathscr{H}_{\eta_+} \longrightarrow \mathscr{H}$$



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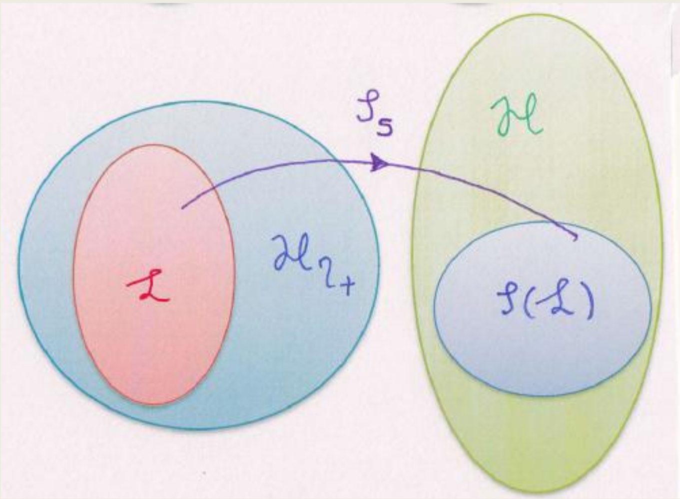

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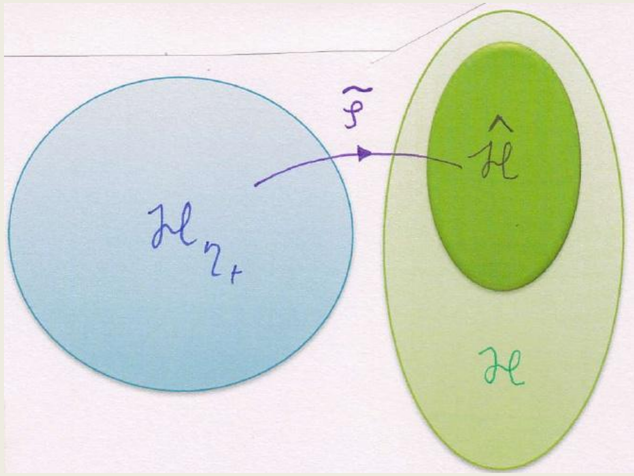

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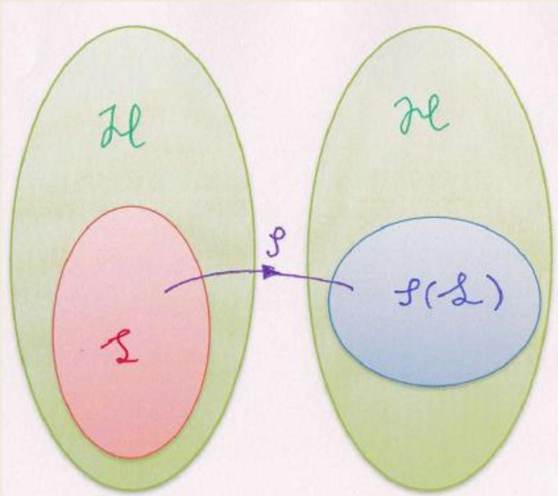
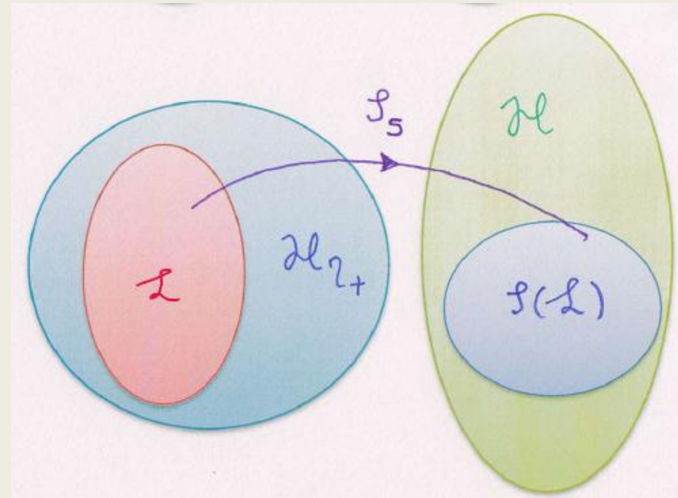


$\rho : \mathscr{H} \longrightarrow \mathscr{H}$



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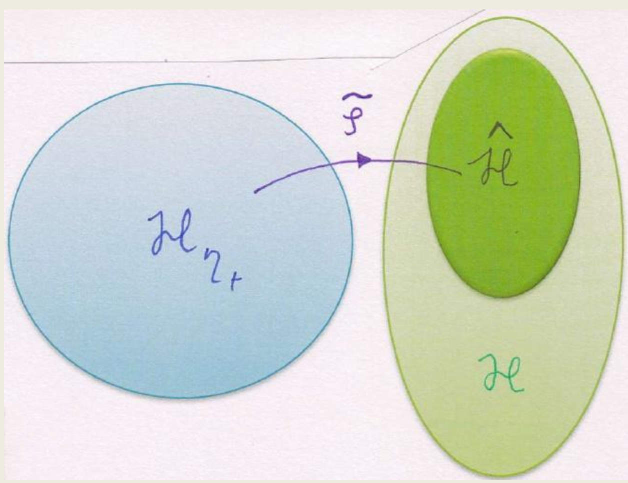

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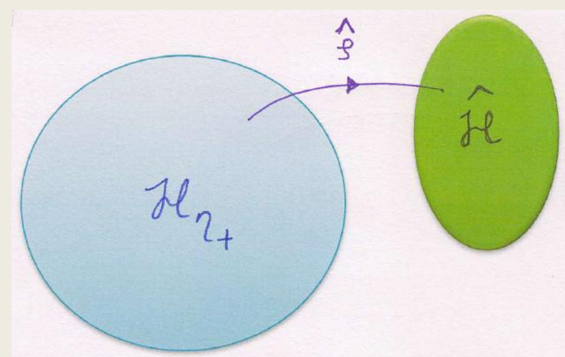

$\hat{\rho}_s : \mathscr{H}_{\eta_+} \longrightarrow \hat{\mathscr{H}}$

"Important Scientific discoveries go through three phases: first they are completely ignored, then they are violently attacked, and finally they are brushed aside as well-known."

Konrad Lorenz
Animal behaviourist
Nobel laureate