Exceptional points behind eigenspace and eigenvalue anholonomies of bound states

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Context

- Adiabatic time evolution of quantum systems

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- Closed cycles

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- Anholonomies

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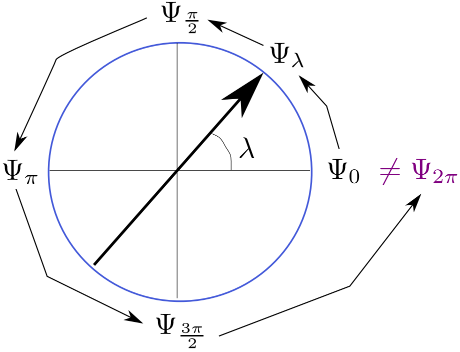
- Adiabatic time evolution of quantum systems
- Closed cycles
- Bound states
- Anholonomies
- ...
- Exceptional points and anholonomies

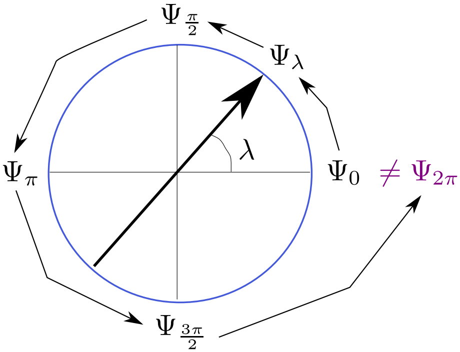
What do we mean by an anholonomy

... a discrepancy between the initial and final “objects” of a cycle.



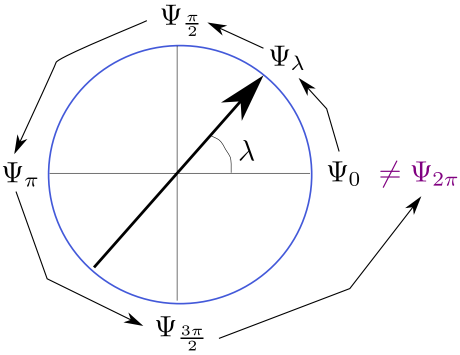
Berry's phase anholonomy (1984)

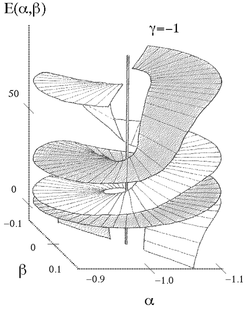
The phase of an eigenstate can exhibits an anholonomy for a periodic change of a environmental parameter. This anholonomy has an established interpretation terms of differential geometry so that we may call it the phase holonomy (Simon 1983).



Eigenspace and eigenvalue anholonomies (Cheon 1998)

About ten years after Berry's work, it was shown that the anholonomies of **eigenspaces** as well as **eigenenergies** of bound states (a.k.a. exotic quantum holonomies) can occur.





from Cheon, PLA **248** (1998)

Aim of this talk

An interplay of **Cheon's eigenspace and eigenvalue anholonomies** of bound states and exceptional points will be explained.

Contents

1. Abundance of the anholonomies in unitary operators
   (AT and M. Miyamoto (Waseda))
2. A gauge theory for the phase and eigenspace anholonomies
   (T. Cheon (Kochi) and AT)
3. Hidden exceptional points underlying the anholonomies
   (S.W. Kim (Pusan), T. Cheon and AT)

Part I: Eigenvalue and eigenspace anholonomies in unitary operators

M. Miyamoto and AT, PRA **76**, 042115 (2007)

Cheon's anholonomies in Floquet operators

AT and M. Miyamoto, PRL **98**, 160407 (2007)

Quasienergy anholonomy and its application to adiabatic quantum state manipulation

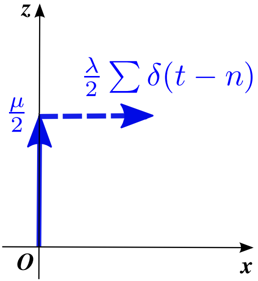
The minimal model: a kicked spin-$\frac{1}{2}$

Let us examine a periodically driven spin-$\frac{1}{2}$:

$$H(t) = \mu \frac{1+\sigma_z}{2} + \lambda \frac{1+\sigma_x}{2} \sum_{n=-\infty}^{\infty} \delta(t-n),$$

where

- μ is the energy gap of the unperturbed system ($\lambda = 0$)
- λ is the adiabatic parameter (the perturbation strength).
- The whole energy is also periodically changed.



$$E_{\rm g}(t) = \frac{\mu}{2} + \frac{\lambda}{2} \sum \delta(t-n)$$

Floquet operator of the kicked spin

The Floquet operator for the unit time interval $-\frac{1}{2} \le t \le \frac{1}{2}$ is

$$U(\lambda) = e^{-i\mu P_z/2} e^{-i\lambda P_x} e^{-i\mu P_z/2}, \quad \text{where} \quad P_{z,x} \equiv \frac{1+\sigma_{z,x}}{2}.$$

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For $U(\lambda)$, the path $C \equiv \{\lambda \mid 0 \le \lambda \le 2\pi\}$ is a closed cycle.

Note that P_x is a projection (i.e., $P_x^2 = P_x$). Hence we have

$$e^{-i\lambda P_x} = (1 - P_x) + e^{-i\lambda} P_x,$$

which implies that $U(\lambda)$ is 2π-periodic in λ.

Eigenvalue problem of $U(\lambda)$

The eigenvalue equation $(n = 0, 1)$:

$$U(\lambda)|n(\lambda)\rangle = e^{-iE_n(\lambda)}|n(\lambda)\rangle, \quad \text{where } E_n(\lambda) \text{ is a quasienergy.}$$

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The solution for the case $\mu=\pi$:

$$E_n(\lambda) = n\pi + \frac{\lambda}{2}, \quad \text{or,} \quad z_n(\lambda)\left(\equiv e^{-iE_n(\lambda)}\right) = (-1)^n e^{-i\lambda/2},$$

$$|0(\lambda)\rangle = \cos\frac{\lambda}{4}|0\rangle + \sin\frac{\lambda}{4}|1\rangle, \quad |1(\lambda)\rangle = -\sin\frac{\lambda}{4}|0\rangle + \cos\frac{\lambda}{4}|1\rangle.$$

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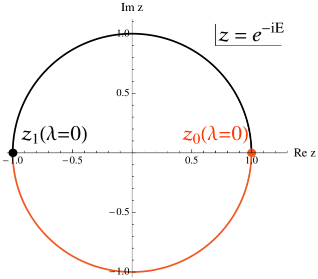
cf. $U(\lambda + 2\pi) = U(\lambda)$.

Eigenvalue and eigenspace anholonomies

The eigenvalue $z_n(\lambda) = (-1)^n e^{-i\lambda/2}$ is not 2π-periodic, although $U(\lambda)$ as well as the spectrum set $\{z_0(\lambda), z_1(\lambda)\}$ has a period 2π.



After the completion of the cycle C, the interchange of eigenvalues occurs.

Eigenvalue and eigenspace anholonomies

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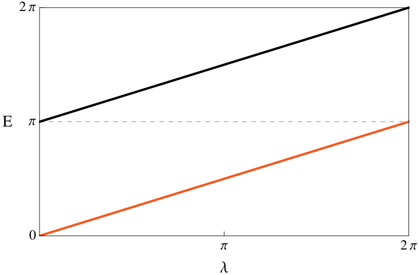

The quasienergies has Möbius strip-like structure.

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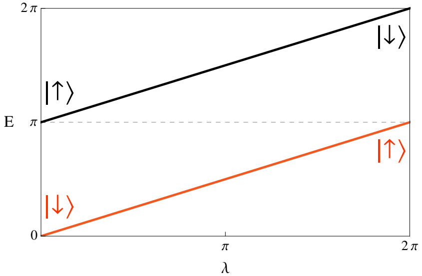


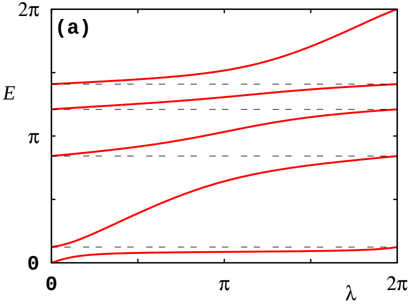
The adiabatic cycle C also induces another anholonomy in the eigenspace: e.g., $|0(\lambda = 2\pi)\rangle = |1(\lambda = 0)\rangle \quad (\perp |0(\lambda = 0)\rangle\ )$.

A generalization to multiple-level systems

The eigenvalue and eigenspace anholonomies occur in a unitary operator

$$U(\lambda) = U_0 \exp(-i\lambda|v\rangle\langle v|),$$

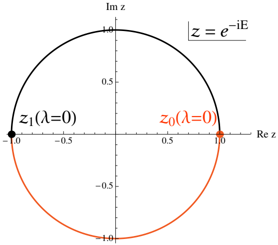
for generic U_0 and $|v\rangle$.

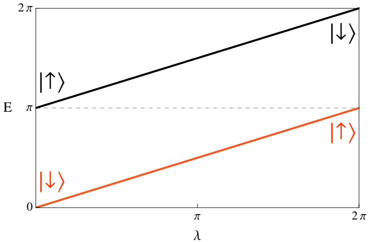


(cf. Combescure 1990, Milek and Seba 1990)

Summary of Part I

Examples of the anholonomies in eigenvalues and eigenspaces in unitary matrices are shown.





Part II: A gauge theory for the phase and eigenspace anholonomies

T. Cheon and AT, EPL **85**, 20001 (2009)

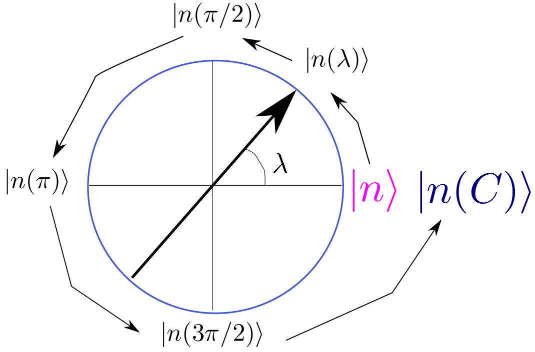
New anatomy of quantum holonomy

AT and T. Cheon, Ann. Phys. **324**, 1340 (2009)

A Unified Theory of Quantum Holonomies

How we characterize the anholonomy in eigenspace?

How do we characterize the discrepancy between the initial eigenstate $|n\rangle$ at $\lambda = 0$ and the final state $|n(C)\rangle$ obtained by the adiabatic time evolution along a closed path C?

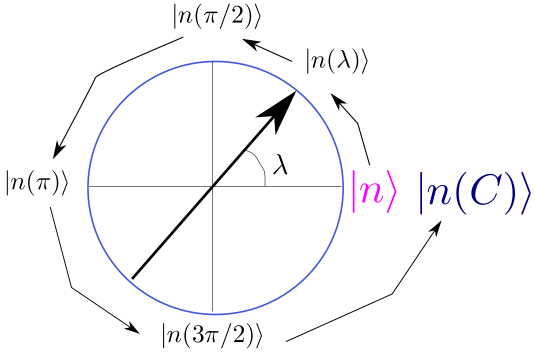


M-matrix

We characterize $|n(C)\rangle$ by the initial basis vectors $\{|m\rangle\}_m$:

$$M_{mn}(C) \sim \langle m|n(C)\rangle$$

which is called M-matrix (or holonomy matrix).



M-matrix: the gauge covariant expression

$$M(C) = \underset{\rightarrow}{\exp}\left(-i\int_C A(\lambda)d\lambda\right)\exp\left(i\int_C A^{\mathrm{D}}(\lambda)d\lambda\right),$$

where

$$A_{mn}(\lambda) \equiv i\langle m(\lambda)|\frac{\partial|n(\lambda)\rangle}{\partial\lambda}, \qquad \text{a non-Abelian gauge connection}$$

and

$$A^{\mathrm{D}}_{mn}(\lambda) \equiv \delta_{mn}A_{nn}(\lambda) \qquad \text{the diagonal part of } A.$$

Remark.
This is an extension of Fujikawa's formulation on the phase anholonomy [Ann. Phys. **322**, 1500 (2007)].

The right factor: adiabatic time evolution of $|n\rangle$

The adiabatic time evolution along C ($\lambda = 0$ to 2π) delivers the initial eigenstate $|n(\lambda = 0)\rangle$ to

$$|n(C)\rangle \equiv |n(\lambda = 2\pi)\rangle \exp\left(i \int_C A_{nn}(\lambda) d\lambda\right) \exp\left(-i \int_0^t E_n(\lambda_\tau) d\tau\right)$$

whose phase factor is governed by Mead-Truhlar-Berry's Abelian gauge connection

$$A_{nn}(\lambda) = i\langle n(\lambda)|\frac{\partial |n(\lambda)\rangle}{\partial \lambda}$$

and the eigenenergy $E_n(\lambda)$.

The left factor: the multi-valuedness of $|n(\lambda)\rangle$

To examine the multi-valuedness in the cycle C, the whole basis vectors

$$f(\lambda) \equiv \left[|0(\lambda)\rangle, \quad |1(\lambda)\rangle, \quad \ldots, \quad |n(\lambda)\rangle, \quad \ldots \right]$$

must be taken into account at a time.

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must be taken into account at a time.

The parametric evolution of $f(\lambda)$ is determined by the non-Abelian gauge connection $A(\lambda)$ (cf. Filip and Sjöqvist 2003):

$$i\frac{\partial}{\partial\lambda} f(\lambda) = f(\lambda)A(\lambda), \quad \text{where} \quad A_{mn}(\lambda) \equiv i\langle m(\lambda)|\frac{\partial|n(\lambda)\rangle}{\partial\lambda}$$

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Hence we obtain

$$f(C) = f(\lambda = 0)\underset{\rightarrow}{\exp}\left(-i\int_C A(\lambda)d\lambda\right),$$

where the second factor is not identity under the presence of the eigenspace anholonomy.

Summary of Part II

Both phase and eigenspace anholonomies along and adiabatic cycle C are described by the holonomy matrix $M(C)$, whose gauge covariant expression is

$$M(C) = \underset{\rightarrow}{\exp}\left(-i\int_C A(\lambda)d\lambda\right)\exp\left(i\int_C A^{\mathrm{D}}(\lambda)d\lambda\right),$$

where

$$A_{mn}(\lambda) \equiv i\langle m(\lambda)|\frac{\partial |n(\lambda)\rangle}{\partial\lambda},$$

is the non-Abelian gauge connection, and $A^{\mathrm{D}}_{mn}(\lambda) = \delta_{mn}A_{nn}(\lambda)$ is its diagonal part.

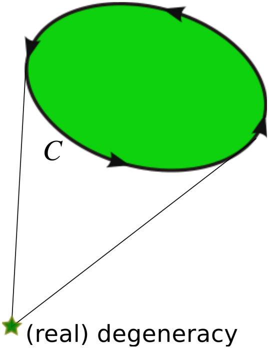
Part III: Hidden exceptional points behind anholonomies of bound states

S. W. Kim, T. Cheon and AT, PLA **374**, 1958 (2010)

Exotic quantum holonomy induced by degeneracy hidden in complex parameter space

Cf. degeneracy behind the phase anholonomy

$$H = B \cdot \boldsymbol{\sigma} \quad \text{(Berry 1984)}$$



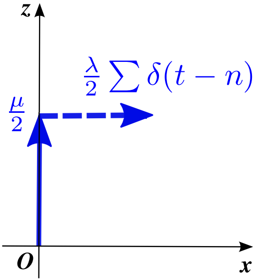
[Geometric phase]

$\propto$ [Solid angle subtended by C at the degeneracy point]

Any similar idea for the exotic anholonomies?

Let us remember our minimal model (we assume $0 < \mu < 2\pi$):

$$H(t) = \mu P_z + \lambda P_x \sum_{n=-\infty}^{\infty} \delta(t-n), \quad \text{where} \quad P_{z,x} \equiv \frac{1+\sigma_{z,x}}{2}.$$



$$E_{\mathrm{g}}(t) = \frac{\mu}{2} + \frac{\lambda}{2}\sum\delta(t-n)$$

The gap of quasienergies $\Delta(\lambda)$

$$\Delta(\lambda) = 2\cos^{-1}\left[\cos\left(\frac{\lambda}{2}\right) \Big/ \cos\left(\frac{i\beta}{2}\right)\right],$$

where $\beta \equiv 2\tanh^{-1}\left(\sin\frac{\mu}{2}\right)$ is strictly positive
($\because$ we assume $0 < \mu < 2\pi$).

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$$\Delta(\lambda) = 2\cos^{-1}\left[\cos\left(\frac{\lambda}{2}\right) \Big/ \cos\left(\frac{i\beta}{2}\right)\right],$$

where $\beta \equiv 2\tanh^{-1}\left(\sin\frac{\mu}{2}\right)$ is strictly positive ($\because$ we assume $0 < \mu < 2\pi$).

The spectral degeneracy occurs only when λ is complexified:

$$\lambda = \pm i\beta + 2\pi k, \qquad k \in \mathbb{Z},$$

or, equivalently,

$$e^{-i\lambda} = e^{\pm\beta}.$$

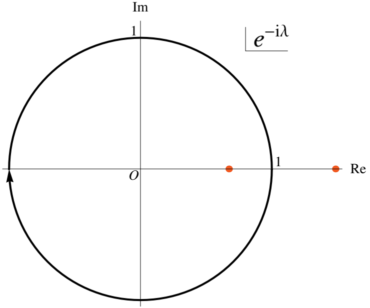

Fig. Configuration of the EPs in $e^{-i\lambda}$-plane.

The degeneracy points are $\sqrt{\ }$-type branch points of $\Delta(\lambda)$.

The eigenvalue anholonomy in terms of EP

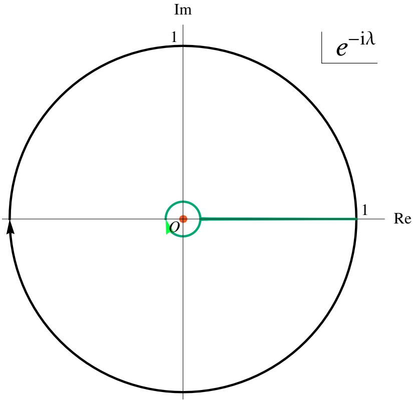

Fig. Configuration of EP, unitary cycle C, and non-Hermitian cycle C' in $e^{-i\lambda}$-plane.

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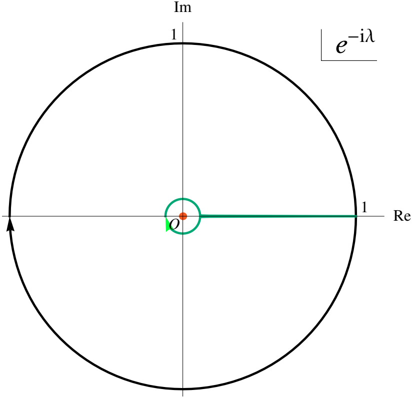

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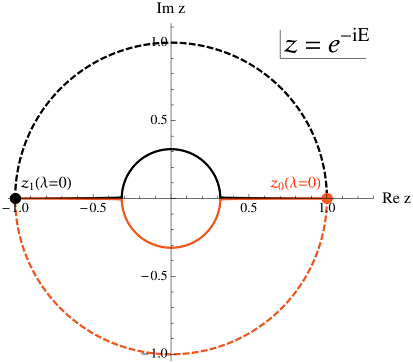


Fig. Parametric evolutions of the eigenvalues along C (dashed) and C' (solid).

Evaluation of $M(C)$ by complex integration

The gauge connection of our example is

$$A(\lambda) = \frac{1}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\frac{\partial\Theta}{\partial\lambda},$$

which satisfies the parallel transport condition $A_{nn}(\lambda) = 0$ in each eigenspace.
Hence, we obtain

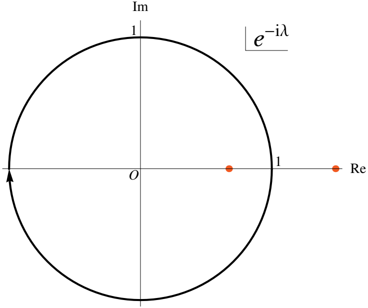
$$M(C) = \exp\left\{-i\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\eta(C)\right\}, \quad \text{where} \quad \eta(C) \equiv \oint_C \frac{1}{2}\frac{\partial\Theta}{\partial\lambda}d\lambda.$$

EP provides the pole of the non-Abelian gauge connection

$$A(\lambda) = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{\partial\Theta}{\partial\lambda}$$

where

$$\frac{\partial\Theta}{\partial\lambda} = \frac{\sinh\beta}{4\sin\frac{\lambda+i\beta}{2}\sin\frac{\lambda-i\beta}{2}}.$$



From the Cauchy theorem, we obtain

$$\eta(C) = \oint_C \frac{1}{2}\frac{\partial\Theta}{\partial\lambda} d\lambda = \frac{\pi}{2}\mathrm{sgn}(\beta).$$

Hence,

$$M(C) = \exp\left\{-i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\eta(C)\right\} = \mathrm{sgn}(\beta)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Summary of Part III

- Through the complexification of the adiabatic parameter λ, we identified EPs that govern Cheon's eigenvalue and eigenspace anholonomies.

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- Through the complexification of the adiabatic parameter λ, we identified EPs that govern Cheon's eigenvalue and eigenspace anholonomies.
- As for the eigenvalues, an EP provides a $\sqrt{\ }$-type branch point.
- As for the eigenspaces, an EP provides a pole of the non-Abelian gauge connection.
- As an application,

$$M(C) = \underset{\rightarrow}{\exp}\left(-i\int_C A(\lambda)d\lambda\right)\underset{\leftarrow}{\exp}\left(i\int_C A^{\mathrm{D}}(\lambda)d\lambda\right)$$

is evaluated by Cauchy's residue theorem.

Summary

I Examples of Cheon's anholonomies in eigenvalues and eigenspaces are shown.

II The phase and eigenspace anholonomies are unified in the holonomy matrix

$$M(C) = \underset{\rightarrow}{\exp}\left(-i\int_C A(\lambda)d\lambda\right)\underset{\leftarrow}{\exp}\left(i\int_C A^{\mathrm{D}}(\lambda)d\lambda\right).$$

III Hidden degeneracies behind Cheon's anholonomies is revealed. The residue of the gauge potential $A(\lambda)$ at the hidden degeneracy point determines $M(C)$.

References

- S. W. Kim, T. Cheon and AT, PLA **374**, 1958 (2010).
- AT, T. Cheon and S.W. Kim, JPA **45**, 335305 (2012) and references therein.