# Crypto-Hermitian theory of quantum catastrophes

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.

PHHQP XI: Non-Hermitian Operators in Quantum Physics

.

**APC, Paris, France**

August 28, 2012

# contents

## I. Introduction: classical catastrophes

# formulated by

## René Frédéric Thom

September 2, 1923 - October 25, 2002

# classical theory in five paragraphs:

.

a. context

.

b. the simplest catastrophe - the fold

.

c. the most useful classical catastrophe - the cusp

.

d. symmetric cusp

.

e. the abstract classical theory

# a. context

**E. C. Zeeman, Catastrophe Theory**

Scientific American, April 1976; pp. 65 - 70, 75 - 83

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**i. in geometry**

= see singularity theory

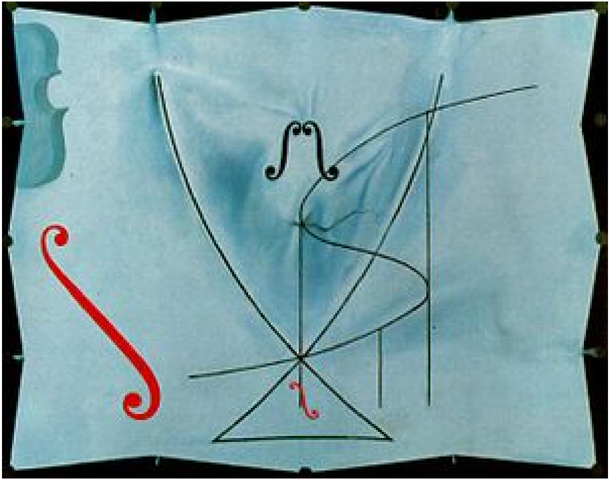
**ii. in nonlinear differential equations**

= see bifurcation theory

**iii. in physics:**

= see the theory of dynamical systems

# see also the Salvador Dalí's last painting (May 1983):



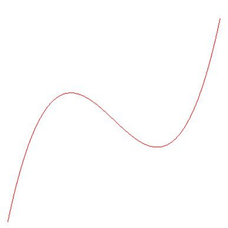
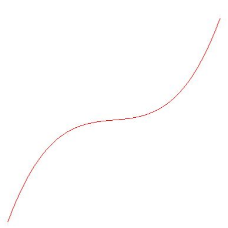
**“The Swallow's Tail — Series on Catastrophes”**

oil on canvas, 73 cm × 92.2 cm, Dalí Theatre and Museum, Figueres

# b. the simplest catastrophe - the fold

**= Lyapunov function $V(x,a) = x^3 + ax$**

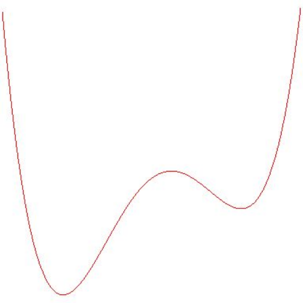
“fold bifurcation”:

 for $a < 0$, and  for $a > 0$.

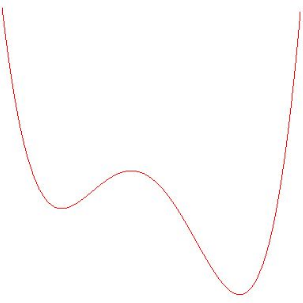
## c. the most useful classical catastrophe - the cusp

**Lyapunov function $V(x, a, b) = x^4 + ax^2 + bx$**

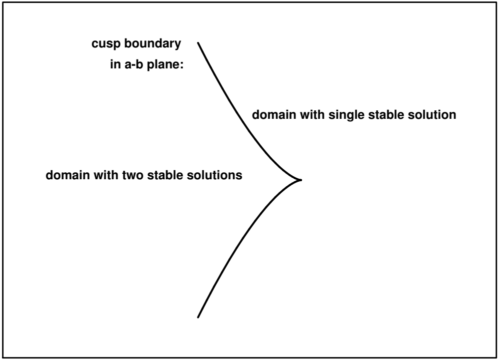
**sign-change of $b$ $\Leftrightarrow$ shape-reflection of $V$**



$\Leftrightarrow$



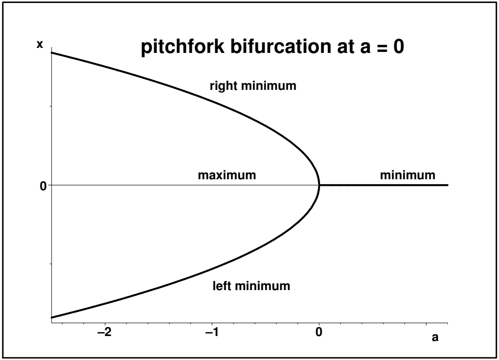
# boundary = “cusp”



# d. symmetric cusp ($b = 0$)

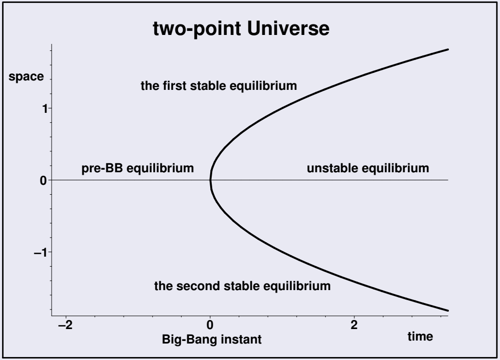
$V(x, a, 0) = x^4 + ax^2$

a collapse in $x - a$ plane



# example: use time $t = -a > 0$

## $V'(x, -t, 0) = 3\,x^3 - 2\,t\,x = 0$ mimics space-time equilibria



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## = purpose: non-equivalent scenarios of time-evolution

- $\boxed{\text{subdomains}}$ of parameters
- their boundaries $\partial \mathcal{D}_s$

# II. The abstract concept of a quantum catastrophe

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- the qualitative theory $\equiv$ GEOMETRY

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## ♡ quantum motion has four aspects:

- $\boxed{\text{time-dependent}}$ eigenvalues $q_n(t)$ and EP at $t = 0$
- $\boxed{\text{time-dependent}}$ wave functions in Hilbert space, $|\psi(t)\rangle \in \mathcal{H}$;
- $\boxed{\text{multiple}}$ observables $F(\vec{\lambda}(t)), G(\vec{\lambda}(t)), H(\vec{\lambda}(t)), \ldots$
- ambiguous Hilbert-space $\boxed{\text{metrics}}$ $\Theta(\vec{\lambda}(t), \vec{\kappa}(t))$.

# obstructions

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2. $\boxed{\text{unfriendliness}}$:

- the friendly "Dirac's" $\Theta = I$ would give

  (1) trivial theory,

  (2) avoided crossings and

  (3) trivial $\partial\mathcal{D}_s = \emptyset$

  (P.T.O.)

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real symmetric matrix :
$$\tilde{\Lambda}^{(4)}(y) = \left[ \begin{array}{cccc} -3 & \sqrt{3}y & 0 & 0 \\ \sqrt{3}y & -1 & 2\,y & 0 \\ 0 & 2\,y & 1 & \sqrt{3}y \\ 0 & 0 & \sqrt{3}y & 3 \end{array} \right]$$

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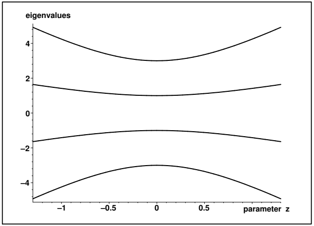
samples the $\boxed{\text{repulsion}}$ of eigenvalues:



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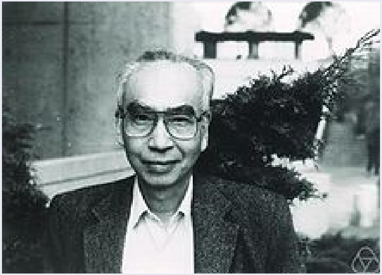
## EPs defined by



**Tosio Kato** (August 25, 1917 - October 2, 1999)
"Perturbation theory of linear operators", Springer, 1966.

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**a. workshops: "The Physics of Exceptional Points"**

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**b. PHHQP talks:**

U. Guenther, D. Heiss, A. Tanaka

# encouragement:

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$\exists$ successful adiabatic versions of q. catastrophes:

MZ, “Quantum Big Bang without fine-tuning in a toy-model”
J. Phys.: Conf. Ser. 343 (2012) 012136 (20 pp.), arXiv: 1105.1282

# III. The benchmark quantum catastrophe: generalized cusp

# example: the initial stage of evolution of the Universe:

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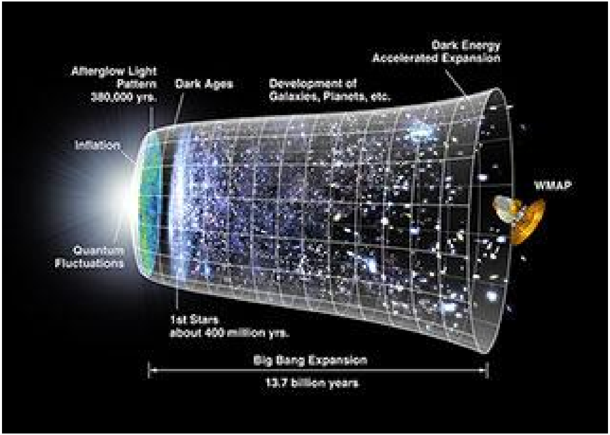

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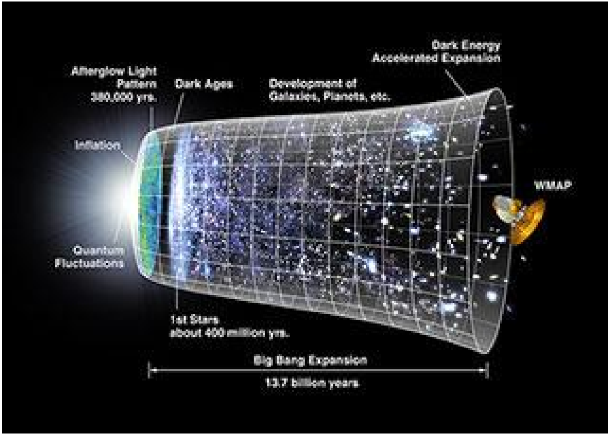


the Thom's Catastrophe Theory must certainly be quantized!

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**formalism: "Three-Hilbert-space formulation of Quantum Mechanics"**

MZ, SIGMA 5 (2009) 001, arXiv:0901.0700

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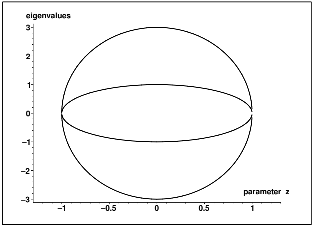
$\boxed{\text{non-Hermitian}}$ matrix $\Longrightarrow$ the $\boxed{\text{attraction}}$ of eigenvalues,



## IIII. BB quantum catastrophe in adiabatic approximation

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## it is worth adding that it took several more years

before Bender *et al* made the idea truly visible among physicists

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the Hermiticity of $\mathfrak{q} = \mathfrak{q}^\dagger$ may be read as the crypto-Hermiticity of $Q = Q^\ddagger = \Theta^{-1}\, Q^\dagger\, \Theta$ where $\Theta = \Omega^\dagger \Omega$ is Hilbert-space metric.

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cf. http://www.sns.ias.edu/~dyson/

**Freeman Dyson** (b. 15. December 1923 in UK)

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## c. require nothing before Big Bang

- spatial grid = $\boxed{\text{complex}}$ before $t = t_{BB} = 0$ (= unobservable)
- full degeneracy: $q_j(t) \to 0$ at $t \to t_{BB} = 0$

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the formalism is known:

**"Time-dependent version of cryptohermitian quantum theory"**

M. Znojil, Phys. Rev. D 78 (2008) 085003 (arXiv:0809.2874v1)

its implementation with $\dot{\Theta} \approx 0$ is straightforward.

# let's start from Big Bang in classical scenario:

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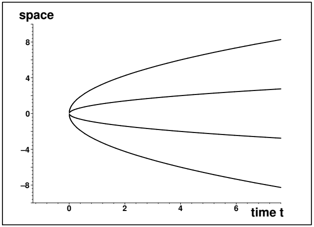


idealized GTR evolution of a discrete $N = 4$ spatial grid

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$$Q^{(2)}_{(a)} = \left[ \begin{array}{cc} 1 & a \\ -a & -1 \end{array} \right], \quad Q^{(4)}_{(a,b)} = \left[ \begin{array}{cccc} 3 & b & 0 & 0 \\ -b & 1 & a & 0 \\ 0 & -a & -1 & b \\ 0 & 0 & -b & -3 \end{array} \right] \quad \ldots$$

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taken from MZ, J. Phys. A: Math. Theor. 40 (2007) 4863 - 4875

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non-numerical construction yields the BB-limit sequence

$$Q_{BB}^{(2)} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad Q_{BB}^{(4)} = \begin{bmatrix} 3 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 1 & 2 & 0 \\ 0 & -2 & -1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & -3 \end{bmatrix} \quad \ldots$$

## let us sample the construction at $N = 2J = 4$:

- take secular equation for $s = E^2$,
  $s^2 + \left(-10 + 2\,b^2 + a^2\right) s + 9 + 6\,b^2 - 9\,a^2 + b^4 = 0$

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e.g., at $N = 8$ we get $D = d^2 = 7$ as a unique root of a seventeenth-degree polynomial (P. T. O.)

$$
\begin{aligned}
&314432\, D^{17} - 5932158016\, D^{16} + 4574211144896\, D^{15} + 3133529909492864\, D^{14} \\
&+9173184951635 61932\, D^{13} + 1675562616489182 75684\, D^{12} + \\
&+146703469297448220 64505\, D^{11} + 720991093724510065469933\, D^{10} + \\
&+62429137451114251409236415\, D^{9} + 6763262782327587843699667 87\, D^{8} + \\
&+405254348029442821531158033 70\, D^{7} + 236197644474644051360524893061 \\
&-14575983663688501214507094831 5366\, D^{5} + \\
&+81299252581229486891579164361 70874\, D^{4} + \\
&-68875673245487669398850290405642067\, D^{3} + \\
&+23532675410182443993680022880 6905073\, D^{2} - \\
&-45376227941462117981555289702 9039797\, D + \\
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- for example, the values of $A = a^2$ are given by the rule $\alpha \times A =$ (a polynomial in $D$ of 16th degree) where the number of digits in the auxiliary integer constant $\alpha$ exceeds one hundred.

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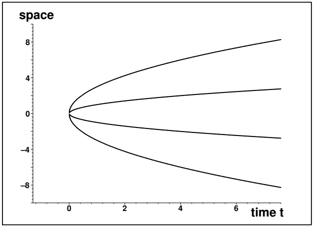


eigenvalues at $N = 4$

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**further details: MZ, "Quantum catastrophes: a case study."**

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a weak point = the adiabaticity assumption

# IIIII. Coriolis-admitting theory and Inflation Period

# standard probabilistic interpretation

## THEOREM 2.

the time-evolution of the system is generated by Hermitian $\mathfrak{h} = \Omega\,\mathrm{H}\,\Omega^{-1}$ which acts on $|\psi\succ = \Omega|\psi\rangle$ as follows,

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The friendly Schrödinger-equation pull-back contains a Coriolis term,

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**details in *loc. cit.***

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see MZ, J. Phys. A: Math. Theor. 40 (2007) 13131-13148

# IIIIII. Discussion

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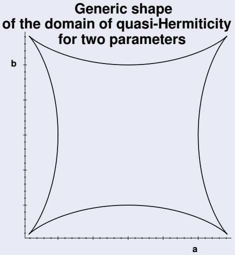
THEOREM 3: $\boxed{\text{near BB, physical domain } \mathcal{D}^{(N)} = \text{a flat layer}}$

# hypersurfaces $\partial\mathcal{D}$ near BB points:

they are all $\boxed{\text{cusp-shaped!}}$ (generic feature)

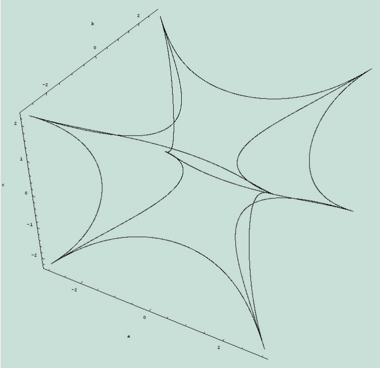
$= \boxed{\text{benchmark}}$

first: two-dimensional quantum catastrophe




cf. MZ, Phys. Lett. B 647 (2007) 225 - 230 (quant-ph/0701232).

# the three-dimensional quantum-cusp surface $\partial\mathcal{D}$



**picture drawn by Petr Siegl in his diploma thesis**

"Quasi-Hermitian Models", FNSPE CTU Prague, 2008

# definition: BB-type quantum catastrophe

$=$ the motion through the EP spike in parametric space $\mathcal{D}$

**realization via the prototype benchmark:**

$A = B = \ldots = 0$, positive $t \equiv r^2$, anti-time $z = \sqrt{1 - r^2}$

**grid points in closed form:**

$(N-1)r, (N-3)r, \ldots, 1, -1, \ldots, -(N-1)r$

**the theory is non-adiabatic: the Coriolis force**

(1) is added to Hamiltonian, $G(t) = H(t) - \Sigma(t)$
(2) may be large, $\Sigma(t) = \mathrm{i}\Theta^{-1}(t)\dot{\Theta}(t)$ is explicit

# the explicit constructions

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illustration: the $N = 2$ case: $\kappa_1 = \kappa_+ = \sin\alpha$, $\kappa_2 = \kappa_- = \cos\alpha$, $0 < \alpha < \pi/2$ (may be also time-dependent, $\alpha = \alpha(r)$);

$$\Theta = \Theta^{(2)}(\alpha) = \begin{bmatrix} 1 + r\cos 2\alpha & -\sqrt{1-r^2} \\ -\sqrt{1-r^2} & 1 - r\cos 2\alpha \end{bmatrix}$$

$=$ diagonal when $r \to 1$ ($\Rightarrow$ the end of "inflation period").

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eigenvalues of any $\Theta^{(N=2)}(\alpha)$ are never equal (= inflation anisotropy),

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at the end of inflation the anisotropy vanishes

$=$ this result is valid at all $N$

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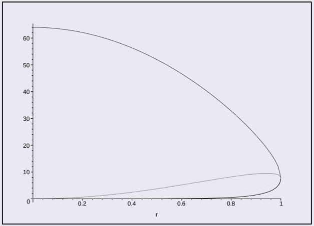


The $N = 4$ sample of the eigenvalues of our metric. The inflation (= the regime of anisotropic metric) ends in a finite time $t = r^2 = 1$.