Non-Hermitian Maxwell Operators, Lasing, Coherent Absorption and PT-symmetric Scattering

A. Douglas Stone Applied Physics, Yale University,
PHHQP, Paris - 8/29/12

1. This talk is 80% non-PT, it is mainly about non-hermitian (NH) *classical* electrodynamics
2. Classical NH wave equations have familiar NH singularities, i.e. poles and EPs – will talk about both, emphasis on pole "motion" (new)
3. Field quantization effects neglected, except laser linewidth, not single photon QM optics
4. Everything is observable despite NH theory
5. Application of NH theory of practical importance – "old" laser theory limited – new phenomena

Non-Hermitian Maxwell Operators, Lasing, Coherent Absorption and PT-symmetric Scattering

Will focus on non-unitary S-matrices, not NH Hamiltonian

Collaborators

Y.-D. Chong – Nanyang

Li Ge – Princeton

Hakan Tureci – Princeton

Stefan Rotter – TU Wien

Arthur Goetschy - Yale

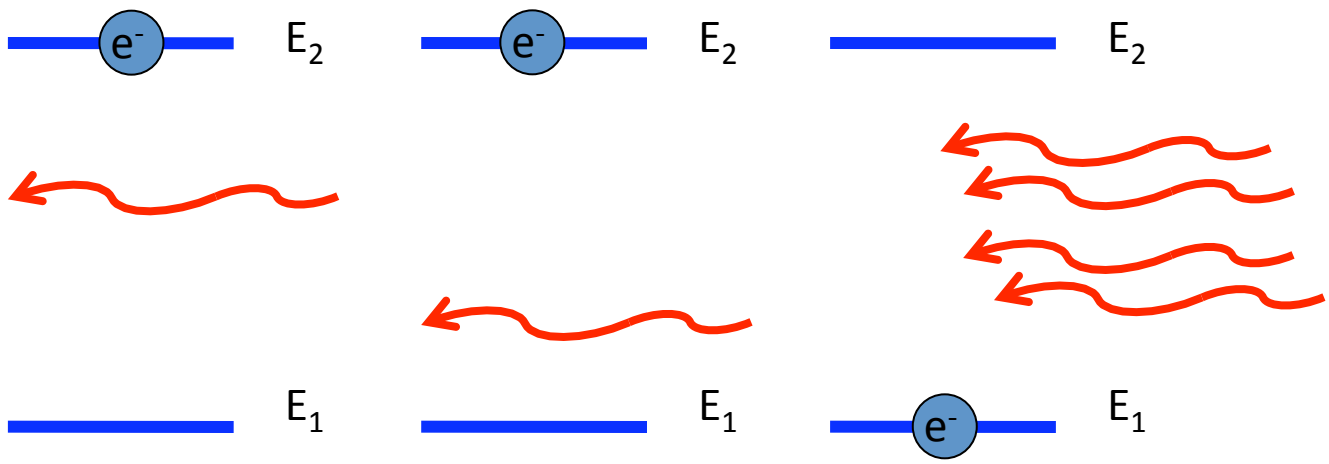
Alex Cerjan - Yale

Hui Cao- Yale (expt)

1. three cases:
   gain => Lasing
   loss => Coherent Absorption
   gain+loss (PT) => Laser-Absorber
2. Laser = nonlinear "NH" wave equation (SALT) – analyzed through linear CF basis set
3. PT-symmetric EM scattering – symmetry-breaking in S-matrix

Principle of the laser:

Pump energy in to create "an inverted state" - now send in light wave



Stimulated emission beats absorption => amplification!

medium has complex amplifying index of refraction:

$$E = E_0 e^{ik(n-in')x} \sim e^{+n'x}$$

Light Amplified by Stimulated Emission of Radiation

It's not a laser yet

A laser is a light source, not an amplifier!
Need to put the atoms in a cavity to trap light

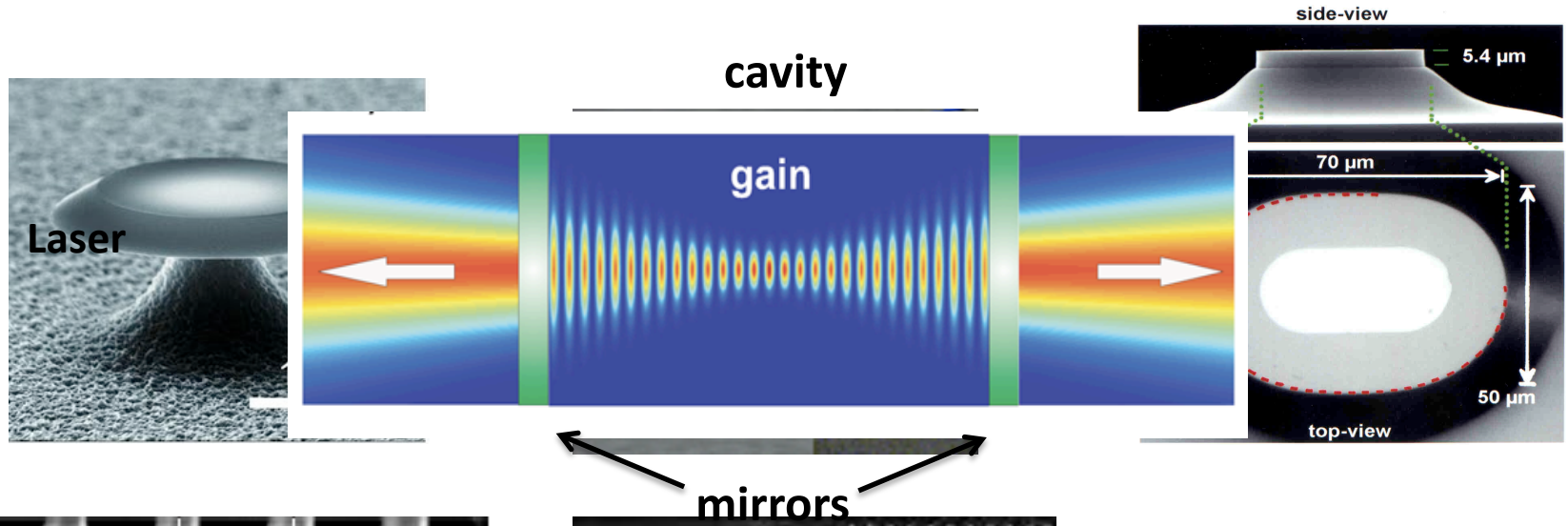
outcoupling loss = gain => $R\, e^{+2n'L} = 1$ => threshold

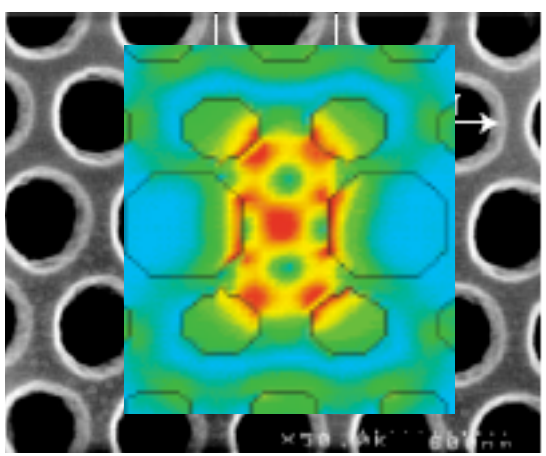
A Laser is self-organized electromagnetic oscillator (feedback), not all modes of cavity contribute to lasing spectrum

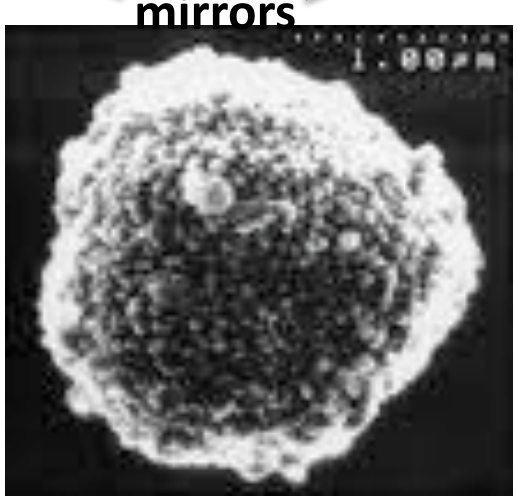
Light Oscillator by Stimulated Emission of Radiation

Modern challenge for laser theory

Complex laser cavities and micro/nano lasers







The strangest one, aggregate of ZnO nanoparticles
No mirrors at all!
=> Random Laser

Openness and complexity: Lasers as scattering systems

$$[\nabla^2 + n^2(\mathbf{r})k^2]\mathbf{E}(\mathbf{r}) = 0$$

$$n(\vec{r}) = \sqrt{\epsilon_c(\vec{r}) + \Delta\epsilon_g(\vec{r})}$$

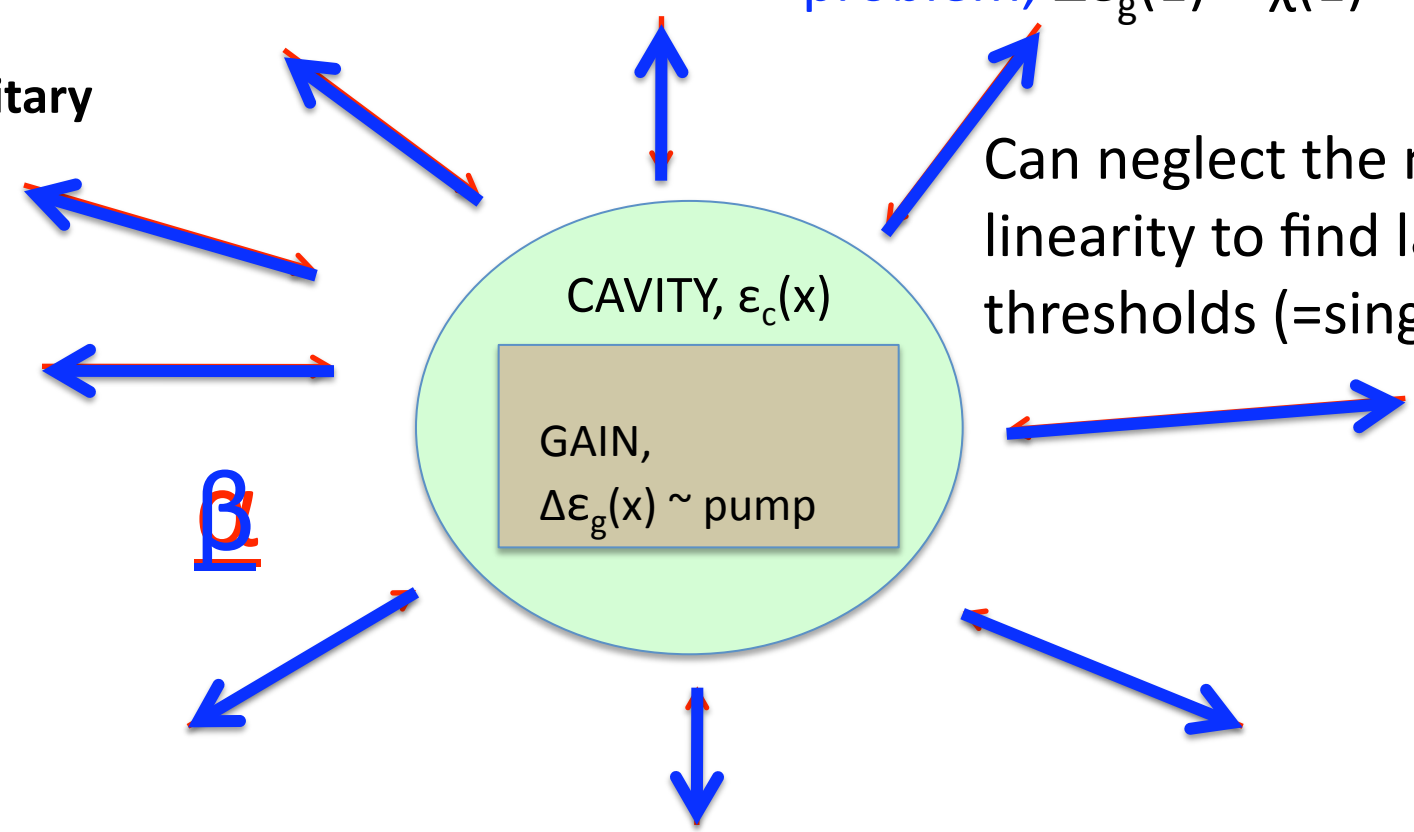
$\Delta\varepsilon_g$,n(r) complex => non-hermitian
for real k, n_2<0 (amplifying)

$$\mathbf{S}(n(\vec{r})k) \cdot \underline{\alpha} = \underline{\beta}$$

amplifying, non-linear scattering problem, $\Delta\varepsilon_g(E) = \chi(E)$

non-unitary

Can neglect the non-linearity to find lasing thresholds (=singularities)



Threshold lasing modes and spectral singularities

$$\mathbf{S}(n(\vec{r})k)\cdot\underline{\alpha}=\underline{\beta}\,,$$

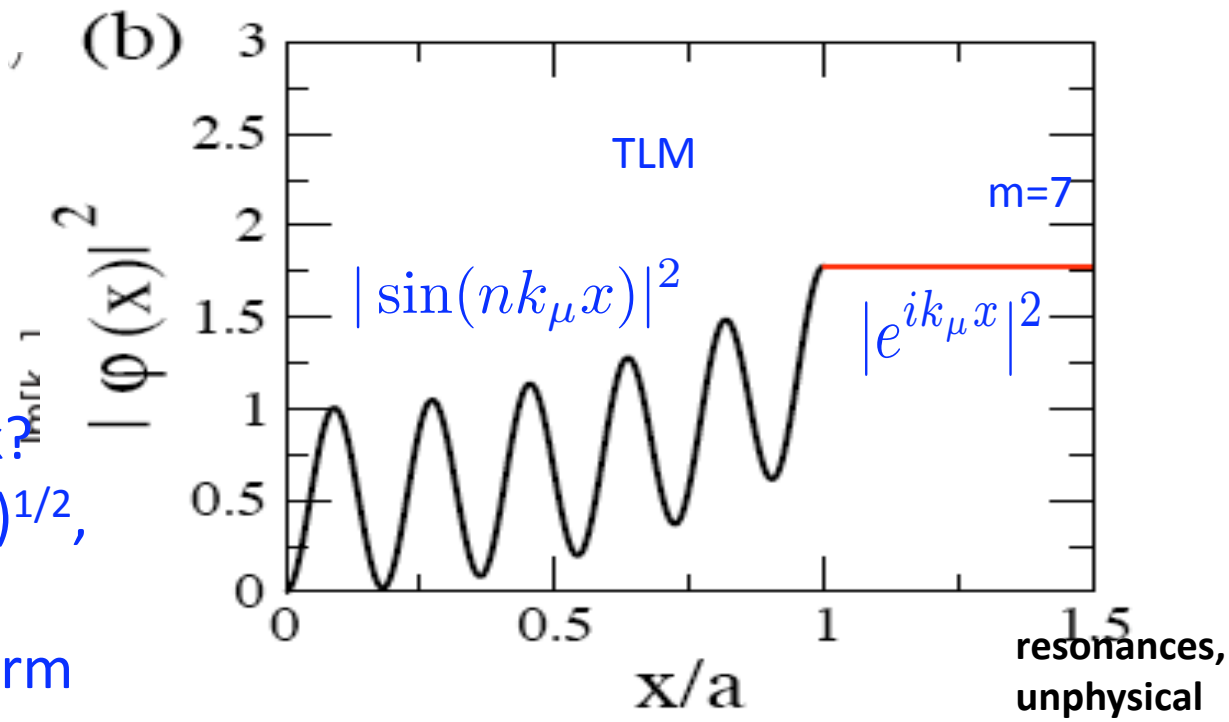
Laser: lasing mode β goes out, nothing in

⇒ S -> ∞ = pole

Can this happen for real k?

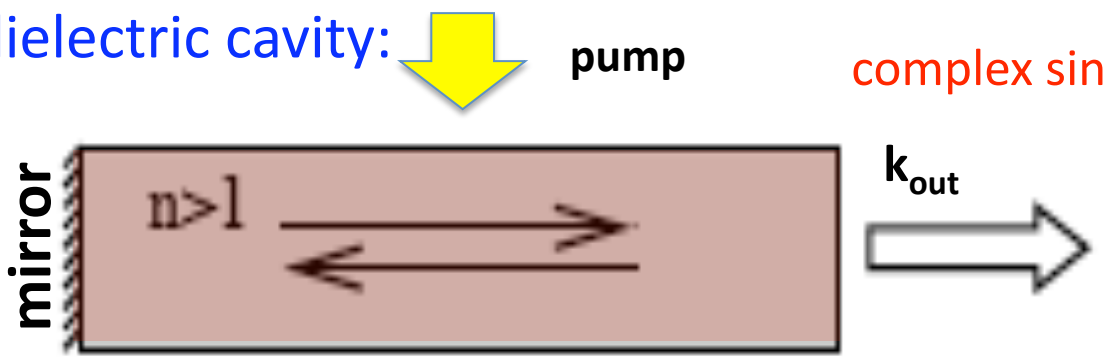
Passive cavity, NO: $n = (\varepsilon_c)^{1/2}$, S unitary, poles complex.

Simple example: 1D uniform dielectric cavity:



resonances, unphysical

complex sine inside, purely outgoing outside



Now add gain medium + pump, $\varepsilon = \varepsilon_c + \Delta\varepsilon_g$

Overall amplitude of TLM will be determined by non-linearity

TLM vs. Resonances/Quasi-modes

$$\varphi_m(x) = \sin(nk_m x)$$

$$\partial_x \varphi|_L = ik\varphi(L)$$

$$(\partial_x^2 + n_m^2 k^2)\varphi_m(x) = 0$$
$$(\partial_x^2 + n_c^2 k_m^2)\varphi_m(x) = 0$$

Threshold lasing modes/constant-flux states

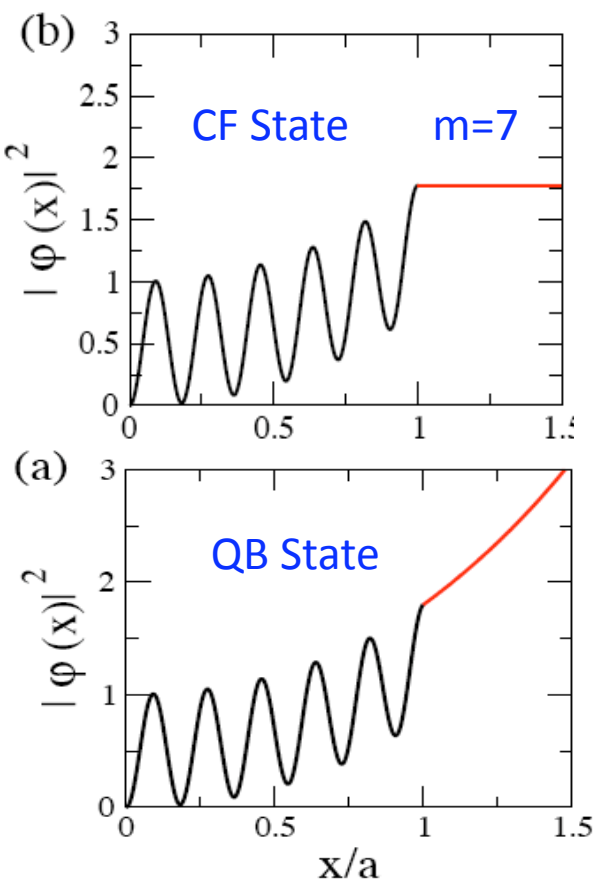
Non-herm BC, k=lasing freq (unknown)

equivalent

$$\partial_x \varphi_m|_L = ik_m\varphi_m(L)$$

$$(\partial_x^2 + n_c^2 k_m^2)\varphi_m(x) = 0$$

resonances/quasi-modes



Resonances/quasimodes ≠ physical solutions of Max. Eqs – describe scattering in passive cavity

TLMs <u>are</u> physical solutions of Maxwell's equations at threshold, for passive cavity plus appropriate gain medium (stabilized by NL)

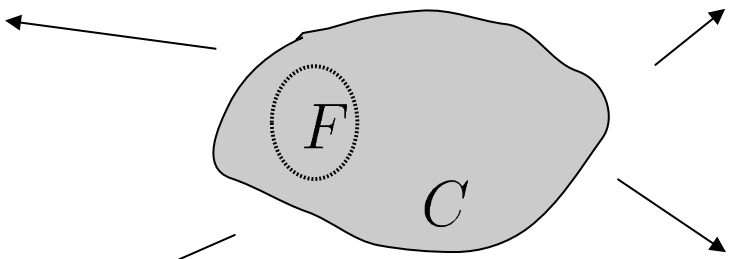
TLMs and Constant-Flux states

Countable set of TLMS, each with specific (k_μ, χ_g^μ, $\phi_\mu(r)$) – not a basis set
To expand arbitrary outgoing function at k_μ
Define non-hermitian basis set of constant-flux CF states

complex CF e-values

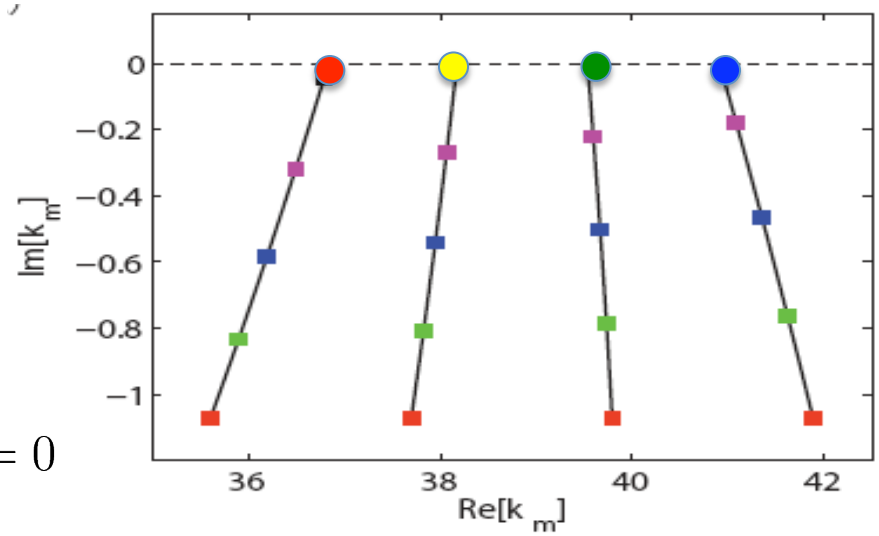
$$\left[\nabla^2 + \left(\epsilon_c(\vec{r}) + \eta_n(k)\, F(\vec{r})\right) k^2\right] u_n(\vec{r}, k) = 0$$

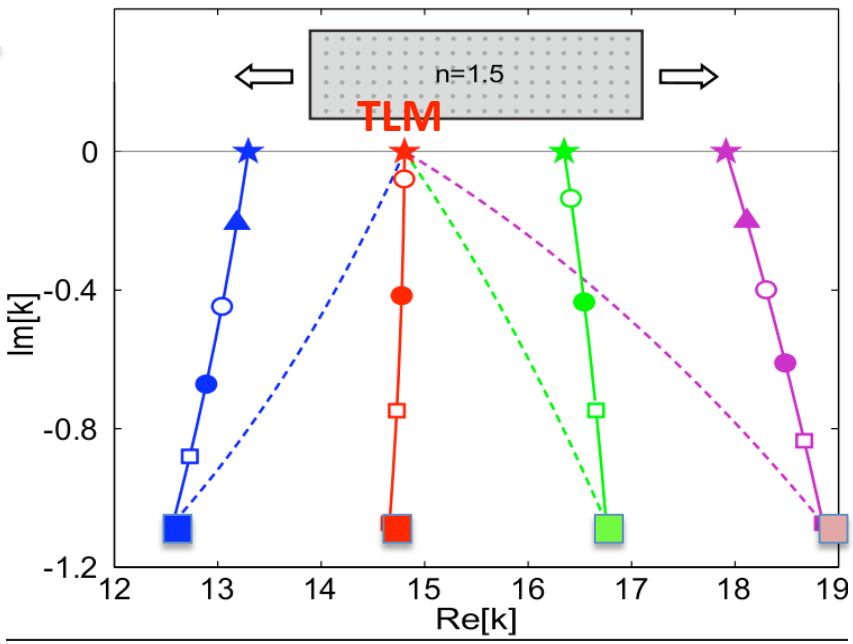
outgoing BC



$$\int_C d^d r\, F(\vec{r})\, u_n(\vec{r}, k)\, u_{n'}(\vec{r}, k) = \delta_{nn'}$$

TLM: $\eta_n(k_\mu) = \chi_g^\mu(k_a, k_\mu, D_0, \ldots)$
Others?
Complete set of biorthogonal fcns
Used to solve non-linear problem
Exceptional pts in CF spectrum (Rotter)



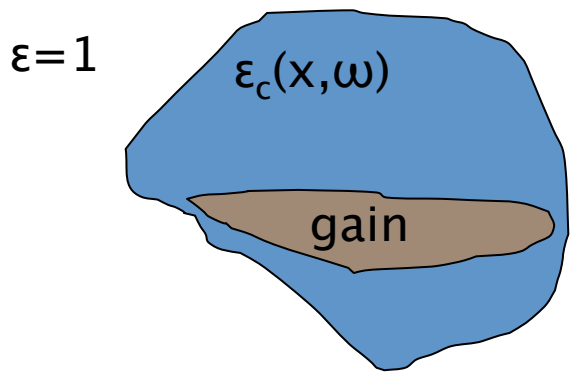


Semiclassical lasing theory

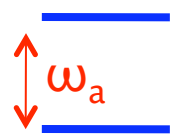
no quant of E,
no spont emission
no laser linewidth

Cavity arbitrary

$$\nabla^2 E(\mathbf{x},t) - \frac{\epsilon_c(\mathbf{x})}{c^2}\frac{\partial^2}{\partial t^2}E(\mathbf{x},t) = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}P_g(\mathbf{x},t)$$



Gain medium:2-level atoms, $\omega_a = ck_a$ (c=1)



$$\boldsymbol{P} = n_a \mathrm{Tr}\, \boldsymbol{p}\rho$$ **Polarization**

$$\boldsymbol{D} = n_a(\rho_{22} - \rho_{11})$$ **Inversion**

Damped, driven, non-linear system

Arbitrary cavity, gain/pumping configuration, openness

Study only steady-state multi-periodic soln (multi-mode lasing)

$$\epsilon_c \ddot{E}^+ = \nabla^2 E^+ - 4\pi \ddot{P}^+$$

$$\dot{P}^+ = (-i\omega_a - \gamma_\perp)P^+ - i|g|^2 E^+ D$$

$$\dot{D} = \gamma_{||}(D_0 - D) + 2i(E^+ P^{+*} - E^{+*} P^+)$$

$$\gamma_\perp = 1/T_2, \quad \gamma_{||} = 1/T_1$$

$$D_0 = pump\ strength$$

$$g^2 = dipole\ coupling$$

Maxwell-Bloch equations
Haken(1963), Lamb (1963)

SALT Outline

- Relation to S-matrix picture: At threshold D => D_0,

$$\dot{P}^+ = (-i\omega_a - \gamma_\perp)P^+ - i|g|^2 E^+ D_0$$

$$[\nabla^2 + (\epsilon_c + \Delta\epsilon_g)k^2]E^+ = 0$$

$$\Delta\epsilon_g \equiv \chi_g = \frac{-iD_0\gamma_\perp}{i(k_a - k_\mu) + \gamma_\perp} \equiv D_0\gamma_\mu$$

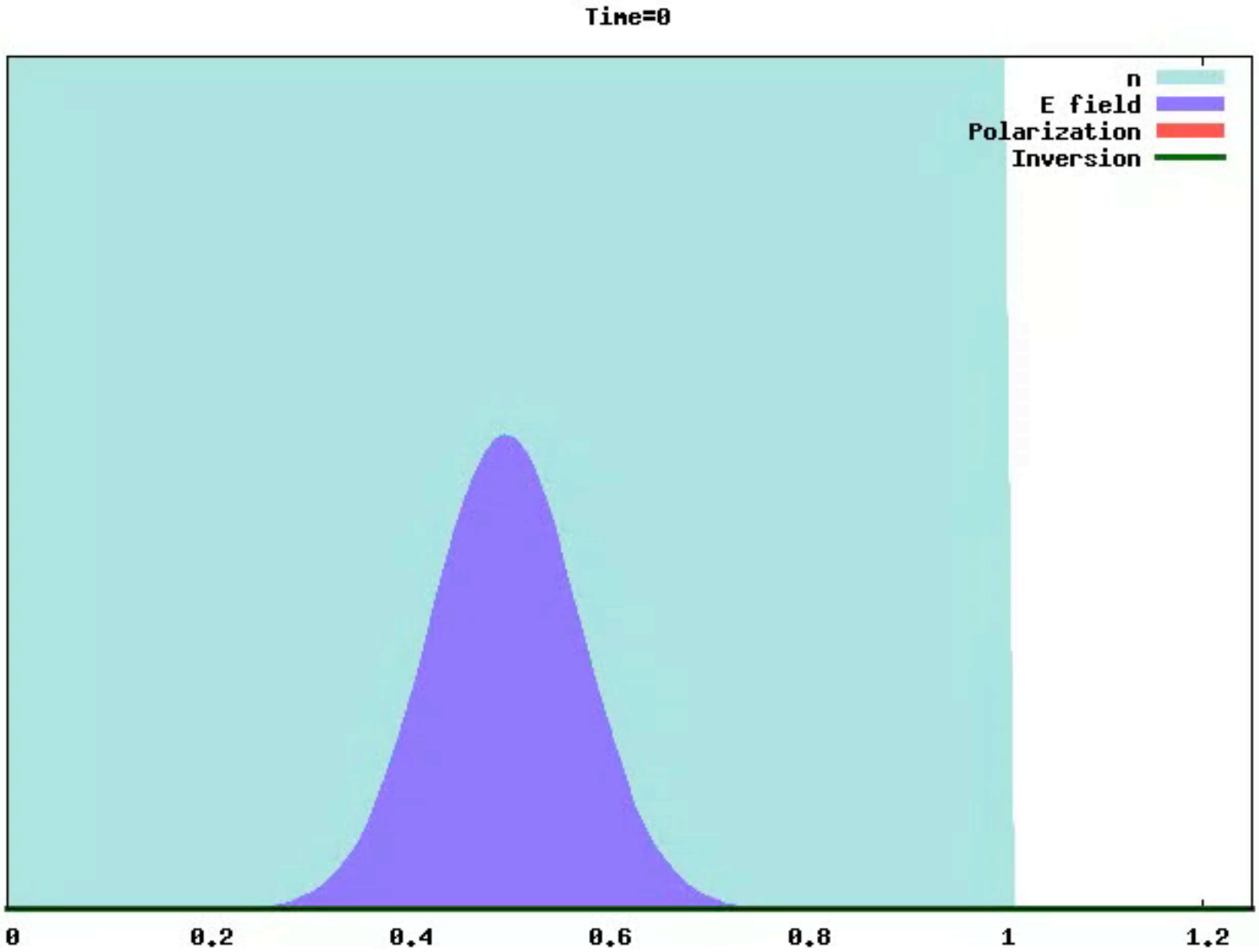
- Above threshold, hole-burning + mode competition, lasing field reacts back on the cavity, inversion eq.

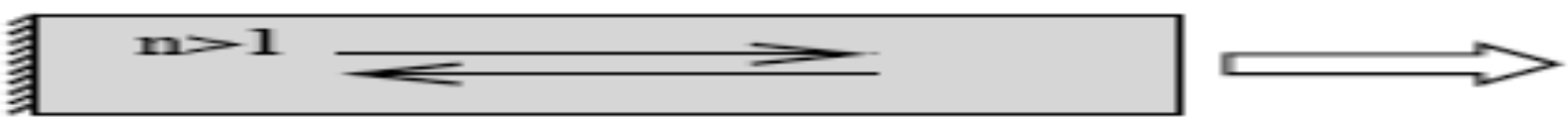
$$\dot{D} = \gamma_{||}(D_0 - D) + 2i(E^+P^{+*} - E^{+*}P^+)$$

- Approximation for multi-mode lasing => dD/dt≈0 (neglects beating of modes).

Valid when: $\gamma_\perp >> \gamma_{||}$

Non-linear Spatial Hole-burning in Laser emission (from integration of MB equations)





Active cavity with laser emission is different from the passive cavity

Solved by SALT (with approx)

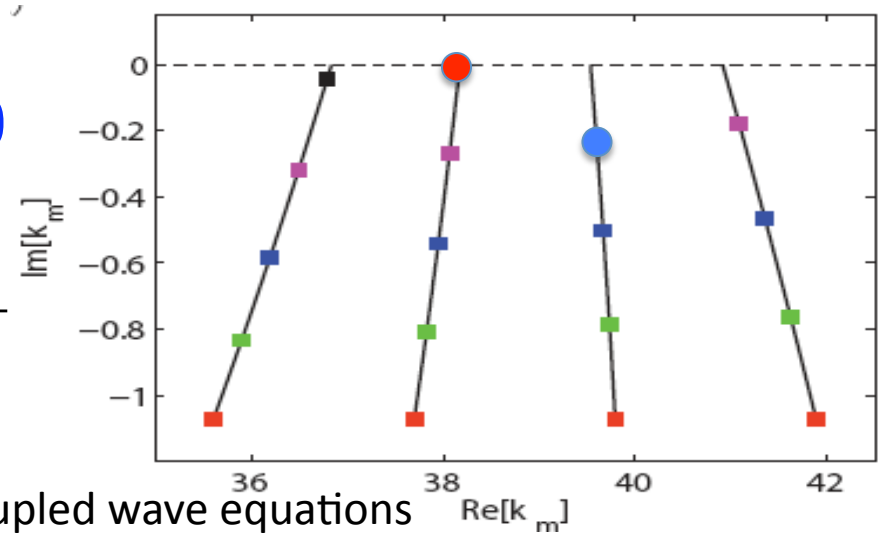
Non-linear SALT equations

At threshold

$$[\nabla^2 + (\epsilon_c + D_0\gamma_\mu)k_\mu^2]E_\mu = 0$$

pump

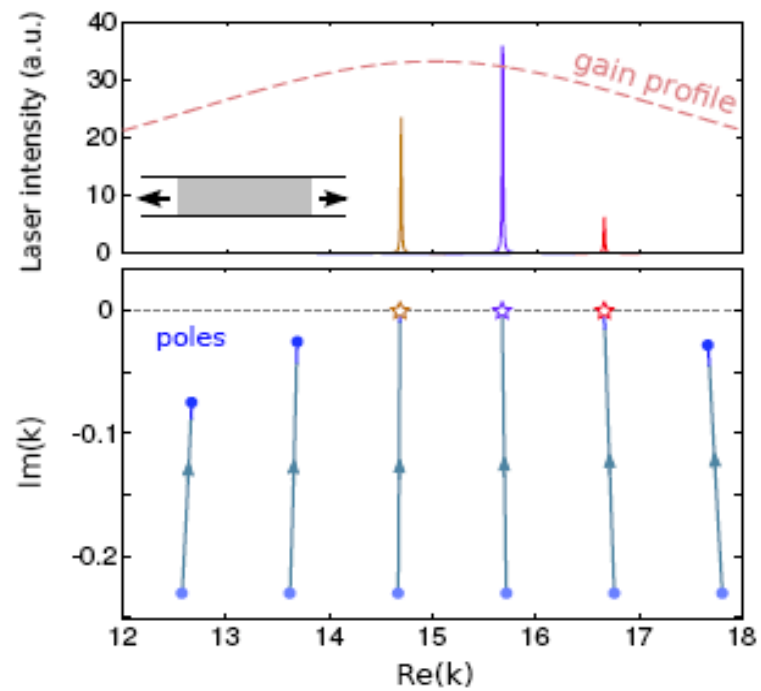
$$\gamma_\mu = \frac{-i\gamma_\perp}{i(k_a - k_\mu) + \gamma_\perp}$$



Above threshold, steady-state, non-linear coupled wave equations

$$[\nabla^2 + (\epsilon_c + \tilde{D}_0\gamma_\mu)k_\mu^2]E_\mu = 0$$

$$\tilde{D}_0 \equiv \frac{D_0}{1 + \sum_\nu^N |\gamma_\nu E_\nu(\mathbf{x})|^2}$$



⇒Steady-state Ab initio Laser Theory:
Thresholds, frequencies, amplitudes, output power, emission patterns, internal fields – noise properties later

SALINE SOLUTION:

$$E(\vec{r},t) = Re[\sum_{\mu}^{N} \Psi_\mu(\vec{r}) e^{-ick_\mu t}]$$

$$\left[\nabla^2 + \left(\epsilon_c(\vec{r}) + \frac{\gamma_\mu D_0 F(\vec{r})}{1+h(\vec{r})}\right) k_\mu^2\right] \Psi_\mu(\vec{r}) = 0$$

$$h(\vec{r}) \equiv \sum_\nu \Gamma_\nu |\Psi(\vec{r})|^2$$

$$E_\mu(r) \longrightarrow \Psi_\mu(r)$$

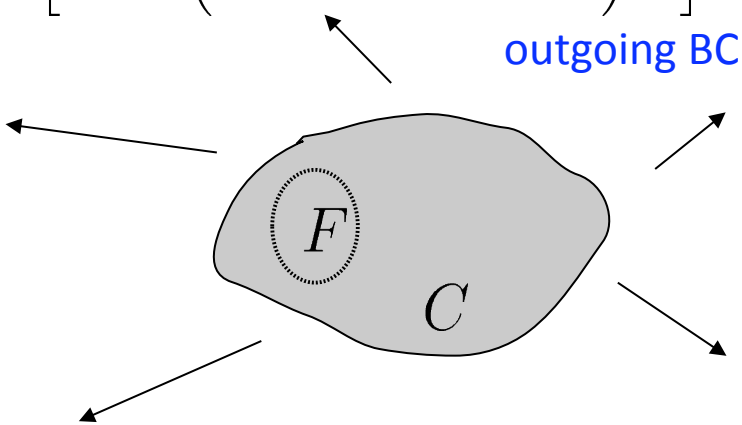
$$\Psi_\mu(\vec{r}) = \sum_n a_n^\mu \, u_n(\vec{r}, k_\mu)$$

Expand soln in basis set of TLMs

Use biorthogonality of $\{u_n\}$

$$\left[\nabla^2 + \left(\epsilon_c(\vec{r}) + \eta_n(k)\, F(\vec{r})\right) k^2\right] u_n(\vec{r}, k) = 0$$

outgoing BC



$$D_0 \sum_{n'} \mathcal{T}_{nn'}^\mu a_{n'}^\mu = a_n^\mu$$

$$\mathcal{T}_{nn'}^\mu \equiv -\frac{\gamma_\mu}{\eta_n} \int_C d^d r \, \frac{F(\vec{r})\, u_n(\vec{r}; k_\mu)\, u_{n'}(\vec{r}; k_\mu)}{1+h(\vec{r})}$$

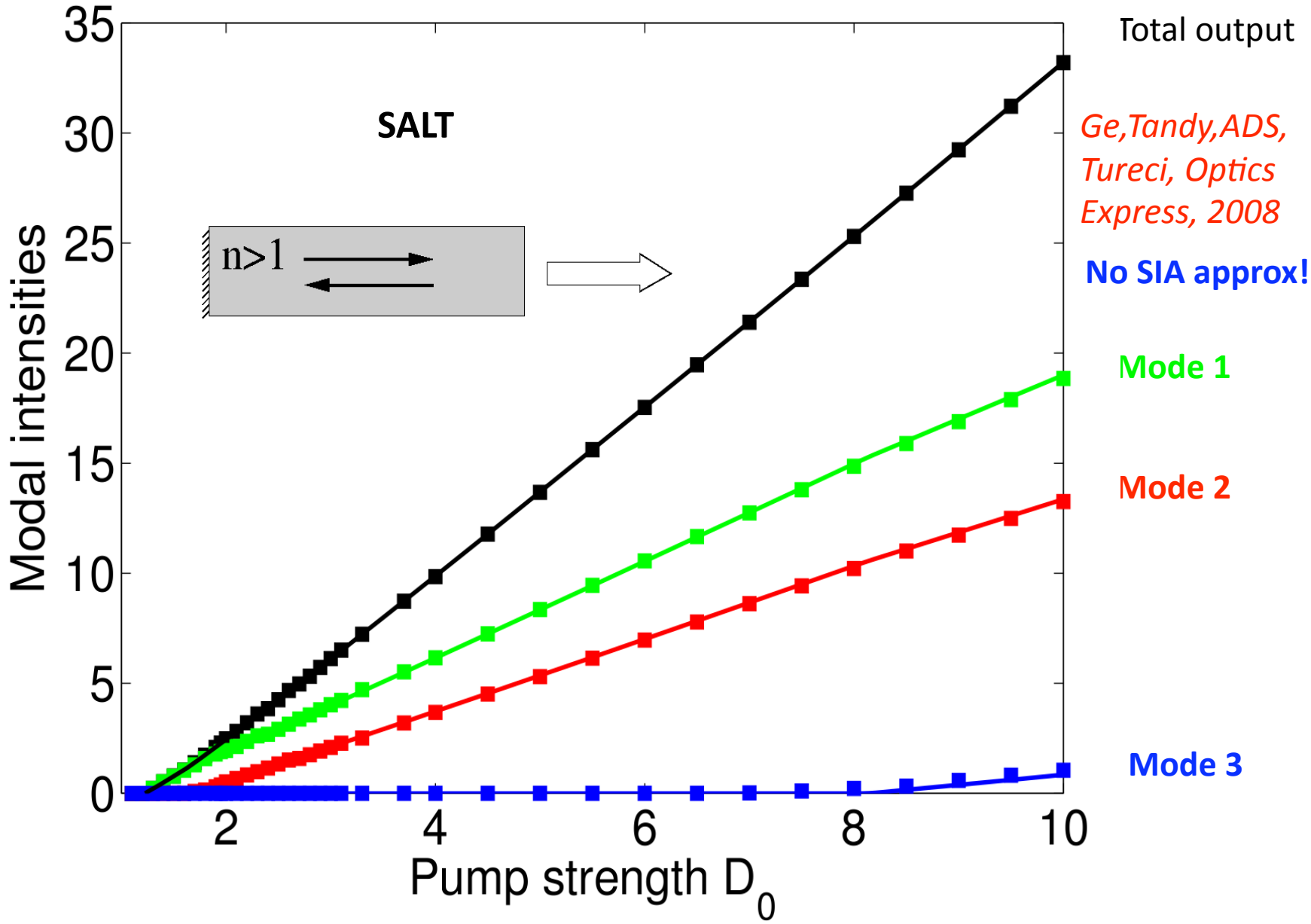
Can also solve non-linear non-hermitian e-value problem directly (Liu and Johnson, MIT)

Truncate sum on CF states,
One term at threshold,
rapidly convergent above

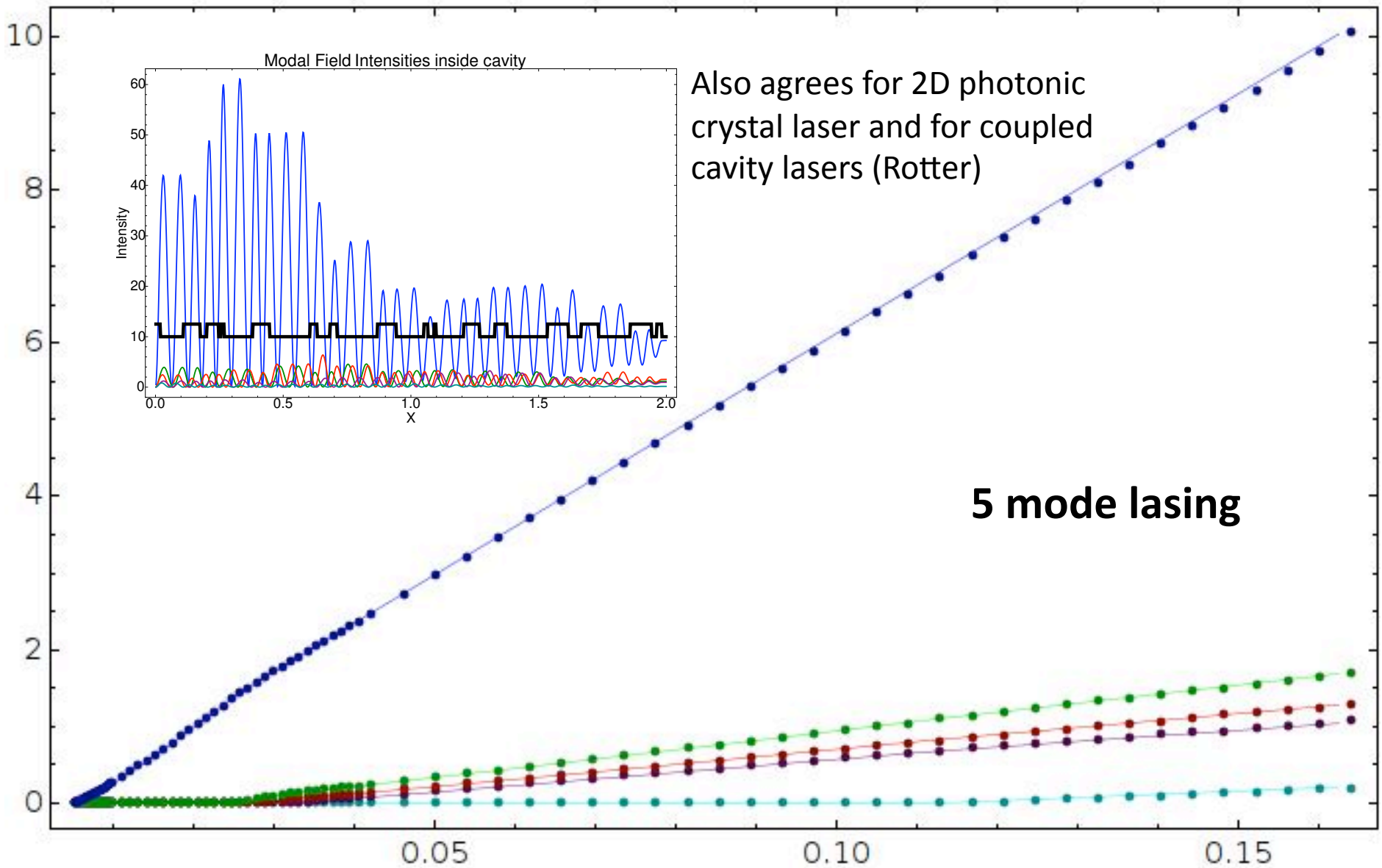
Why SALT is good for you

- ❑ No time integration, orders of magnitude faster
- ❑ Treats openness of the cavity exactly
- ❑ Treats non-linear interactions to infinite order
- ❑ Formulated and applicable to arbitrary cavity
- ❑ How well does it work?

Comparison of SALT and Maxwell-Bloch: intensities



SALT and FDTD agree for 1D random laser



Also agrees for 2D photonic crystal laser and for coupled cavity lasers (Rotter)

5 mode lasing

Laser Linewidth from SALT (Chong, ADS, PRL, 2012)

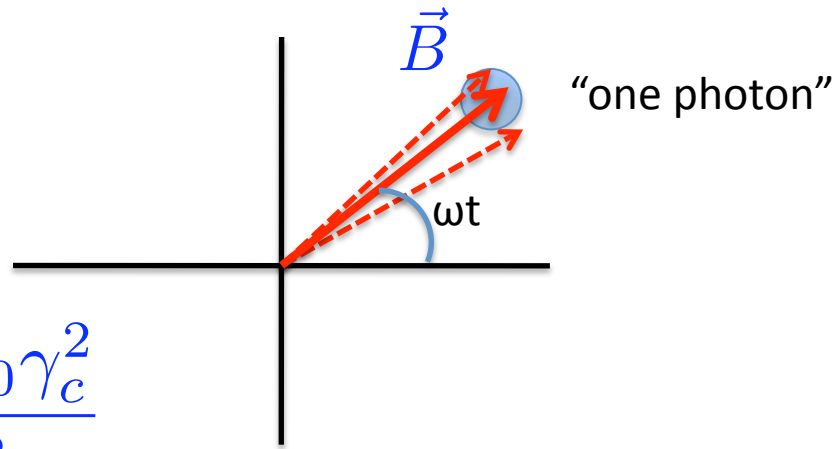
History: Schawlow-Townes, Lax, Petermann, Henry, Haus, Siegman, Woerdman, Goldberg…

$$\Delta\omega_{ST} = \frac{\hbar\omega_0\gamma_c^2}{2P_{out}}$$

Corrected ST formula

Petermann (non-herm)

$$\Delta\omega = K\,|n_B(-T)|\frac{\hbar\omega_0\gamma_c^2}{2P_{out}}$$



Essence of SALT: real lasing pole in S-matrix

diverging e-value at ω_L (Γ_p -> 0)

$$s_0 \approx \frac{\Gamma_L}{(\omega-\omega_L)+i\Gamma_p}$$

$$S \approx |\Psi\rangle \frac{s_0}{\langle\Psi^*|\Psi\rangle} \langle\Psi^*|$$

We know s_0 and $|\psi>$ from SALT!

$$|B_i|^2 = \frac{P}{\hbar\omega_L}|\Psi_L^i|^2$$

Classical output field in channel i, find QM flucts

Generalized laser linewidth

$$b_i(\Omega) = \sum_j S_{ij}(\Omega)\, a_j(\Omega) + \sum_\rho V_{i\rho}(\Omega)\, d^\dagger_\rho(-\Omega).$$

$$b_i(t) = B_i - \frac{\Gamma_L \Psi_L^i}{\Psi_L^T \Psi_L} \int^t dt' e^{-\Gamma_p(t-t')} F(t')$$

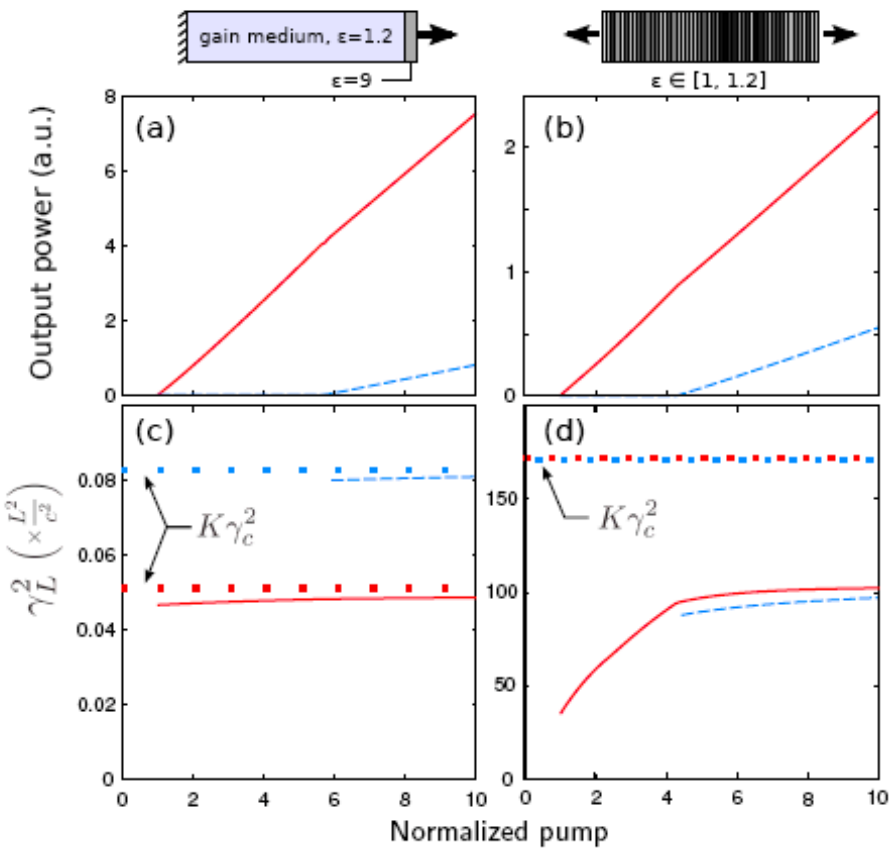
$$\theta_i = \frac{i(b_i^\dagger - b_i)}{2B_i}. \qquad \langle \theta_i(t_1)\theta_i(t_2)\rangle$$

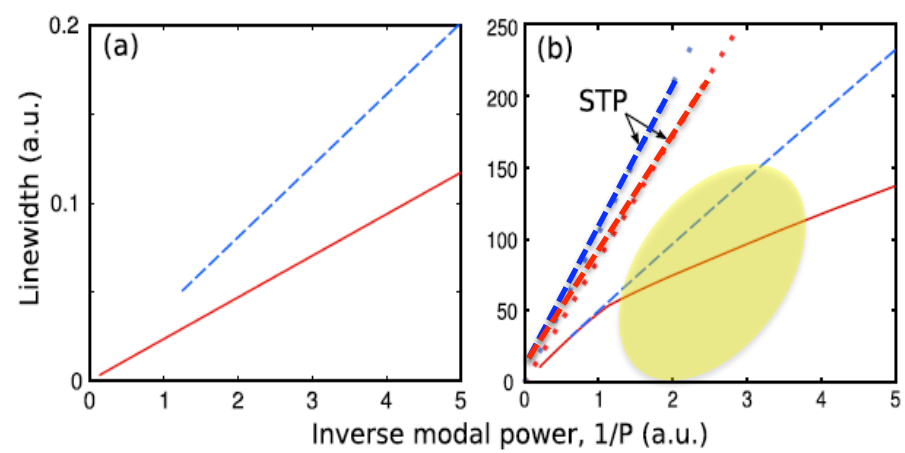
$$\Delta\omega = \frac{|\Gamma_L|^2}{|\Psi_L^T \Psi_L|^2} \frac{\hbar\omega_L}{2P} |n_B|$$

$$\equiv \gamma_L^2 \neq K\gamma_c^2$$

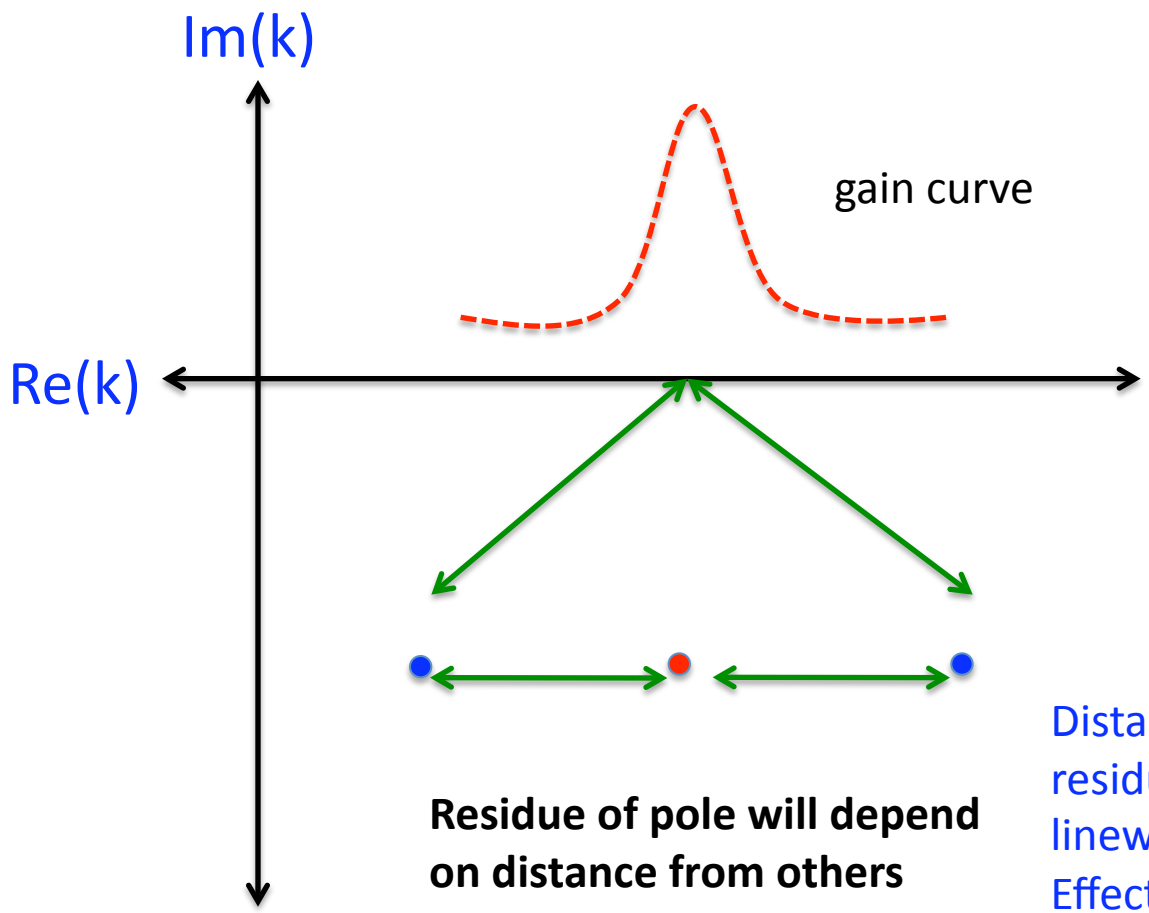
"There is no Petermann factor"!

Violation of 1/P behavior, stems from dispersion of the gain medium





Why are we finding smaller linewidth than STP?



Distance larger,
residue smaller,
linewidth smaller,
Effect of pole motion

Move from poles of S-matrix to zeros (poles of S^{-1})

Coherent perfect absorption: EM S-matrix under time-reversal (CC):

$$\mathbf{S}(n(\vec{r})k)\cdot\underline{\alpha}=\underline{\beta}\quad TR\Rightarrow\ \mathbf{S}(n^*(\vec{r})k)\cdot\underline{\beta}^*=\underline{\alpha}^*$$

Gain => Loss, Incoming ⇔ Outgoing

=> Also an allowed scattering process

LASER: $\underline{\beta}_{out}$ finite, $\underline{\alpha}_{in}$ -> 0 TR-LASER: $\underline{\beta}^*_{in}$ finite, $\underline{\alpha}^*_{out}$ -> 0

Coherent Perfect Absorber (CPA)

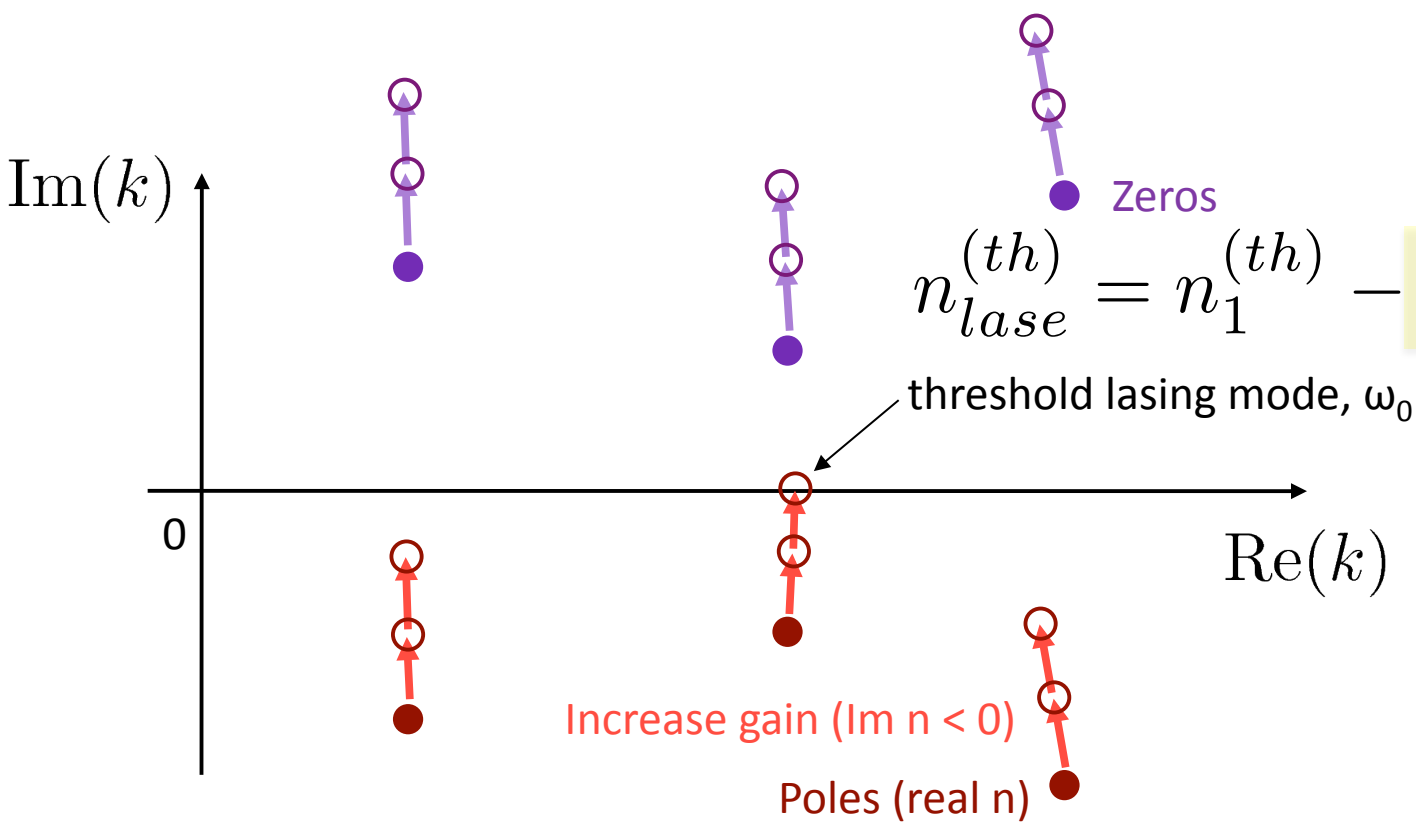
CPA theorem: For every lasing mode at threshold, with frequency ω, and complex (amplifying) index n(r), there exists a CPA mode at ω with complex (lossy) index n*(r)

Chong et al., PRL **105**, 053901 (2010),

Poles and zeros of S-matrix

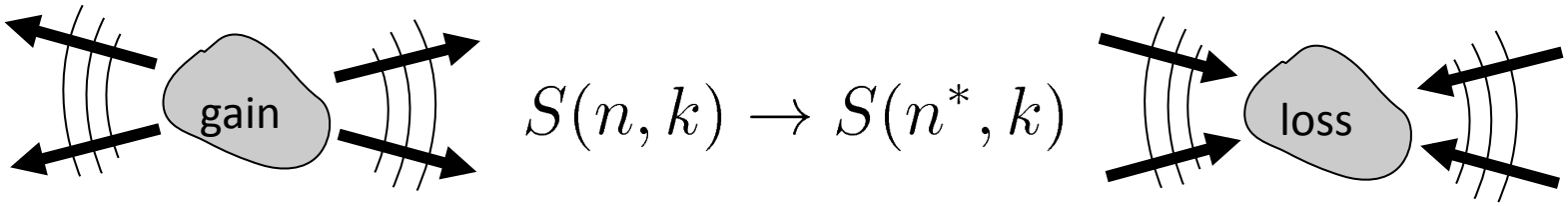
TR symmetry w/o loss or gain

$$k_{pole} = q - i\kappa$$
$$k_{zero} = q + i\kappa$$



$$n_{lase}^{(th)} = n_1^{(th)} - i n_2^{(th)}$$

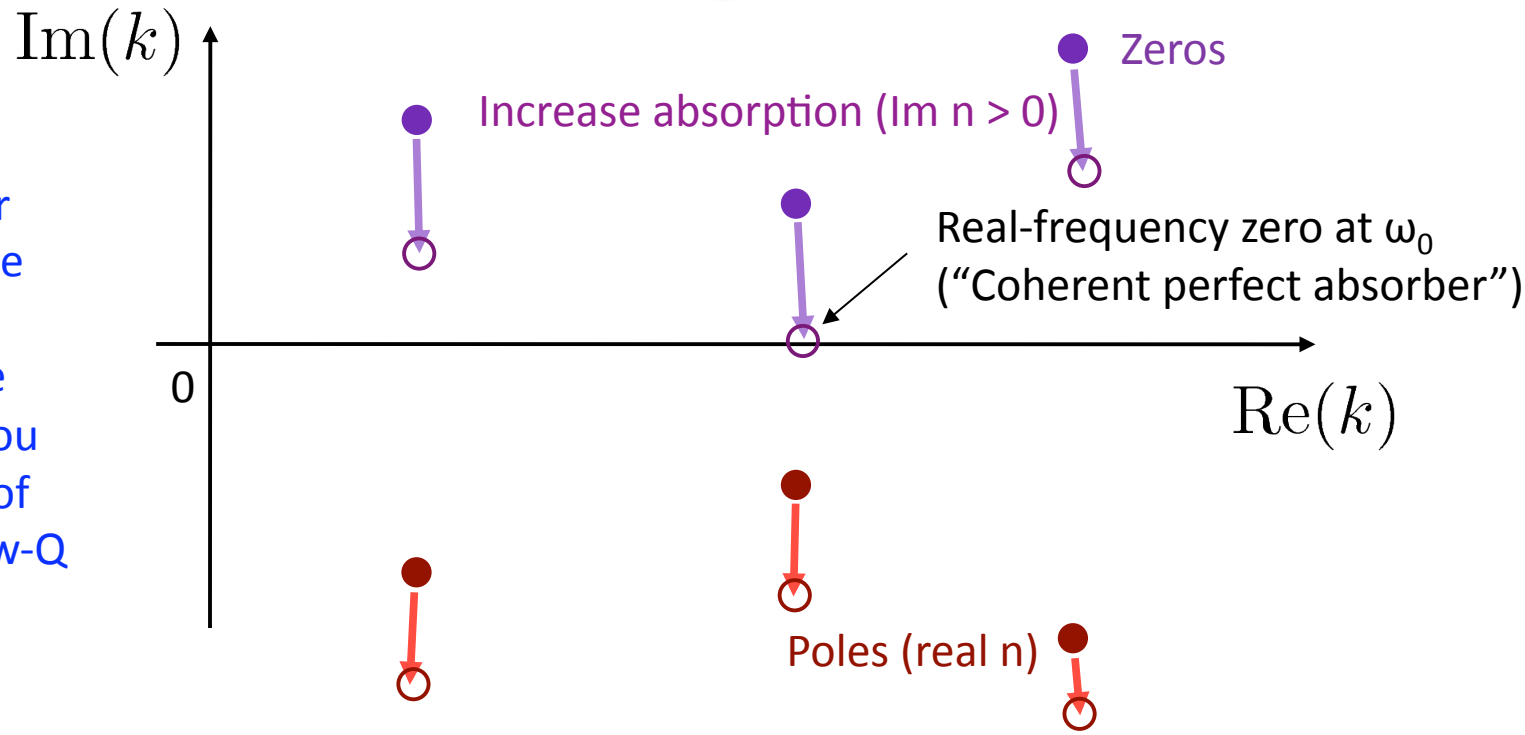
CPA threshold



$$S(n,k) \rightarrow S(n^*,k)$$

$$n_{cpa} = n_1^{(th)} + i n_2^{(th)}$$

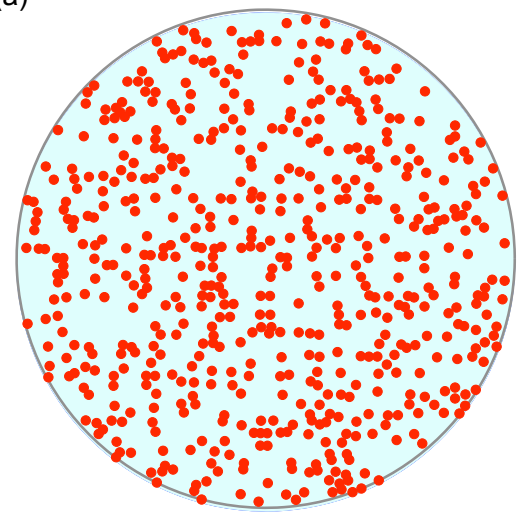
1) Continuity argument – for each resonance there will be a CPA resonance
2) Unlike laser you can access all of them, even low-Q

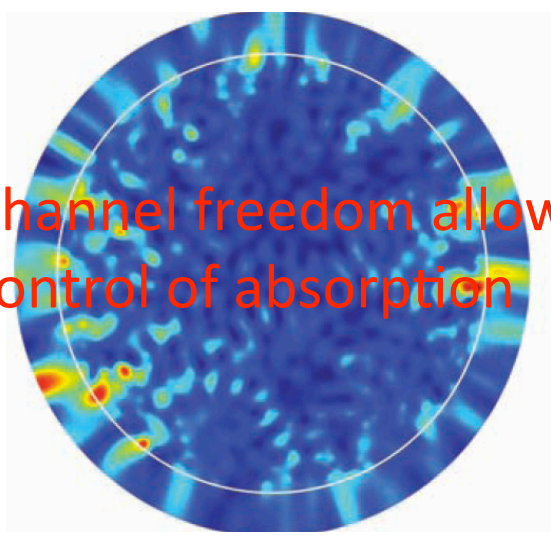


CPA exists in arb. complex system

random/complex "cavity"

(a)

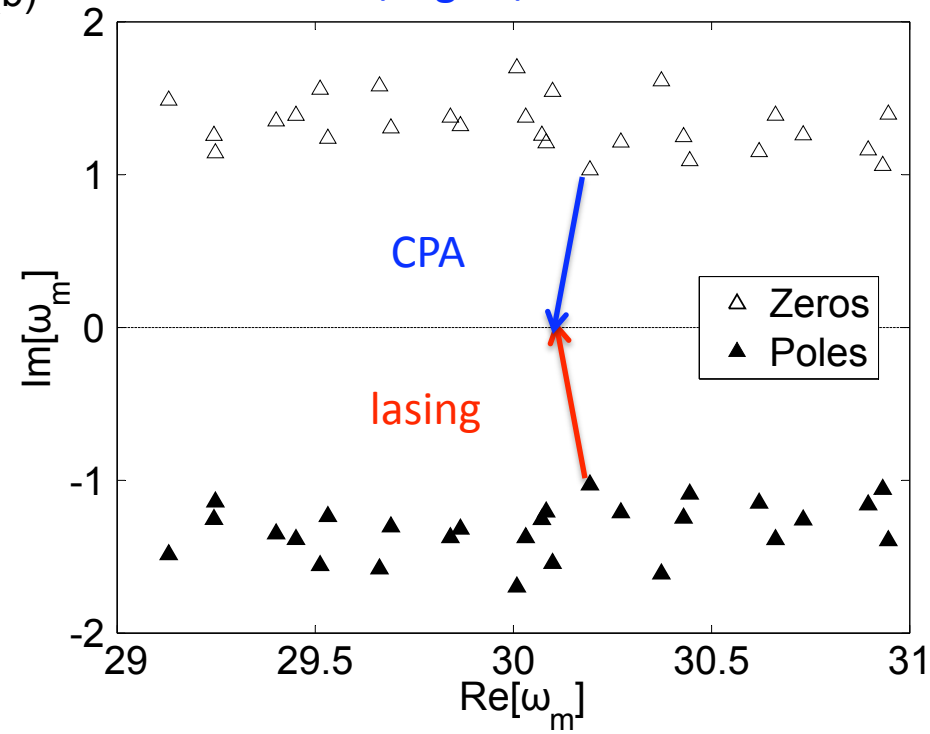




Channel freedom allows coherent control of absorption

(b)

w/o gain/loss



Eigenstates of S-matrix

$$\mathbf{S}(\omega(\vec{n})k)\psi = s\psi$$

- can lase with gain
- Can perfectly absorb with loss

Lasing solns => s ->∞

CPA solns => s -> 0

Other channels non-singular

Was this known?

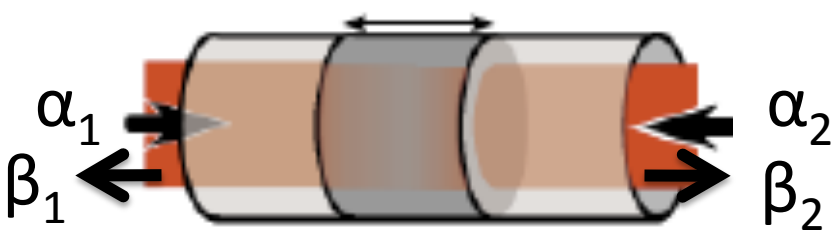
Yes and no

Single channel case well known

Did not use relation to lasing.

1-channel case not sensitive to phase or amplitude of incoming beam

Look at two-channel: simplest one with optical control of absorption

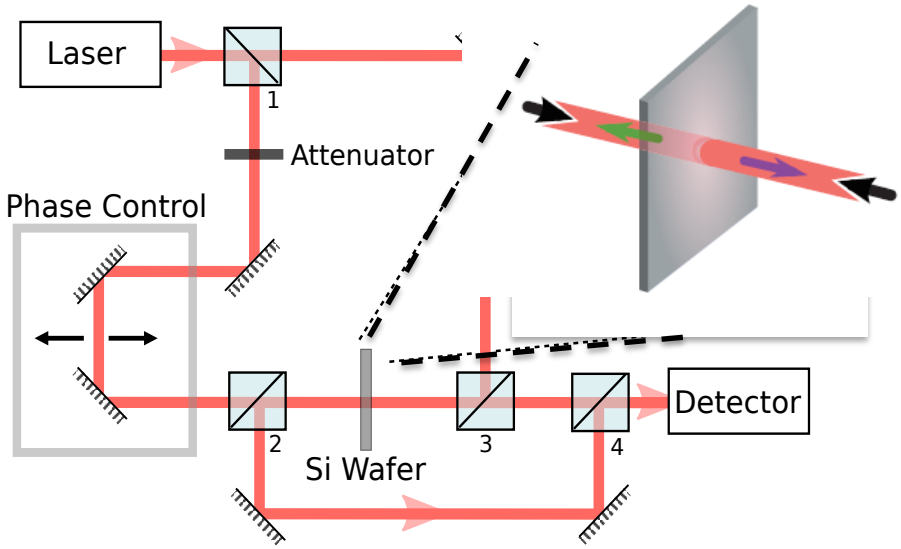


$$S \cdot \alpha = \begin{pmatrix} r_1 & t_2 \\ t_1 & r_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

Absorbing vs. scattering eigenvector

Can demonstrate in intrinsic Si near bandgap (Chong et al., PRL 2010)

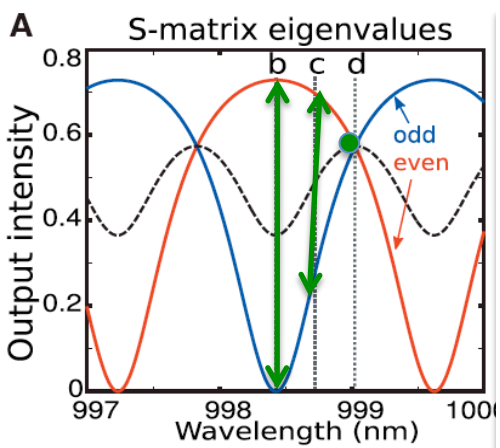
Experimental demonstration



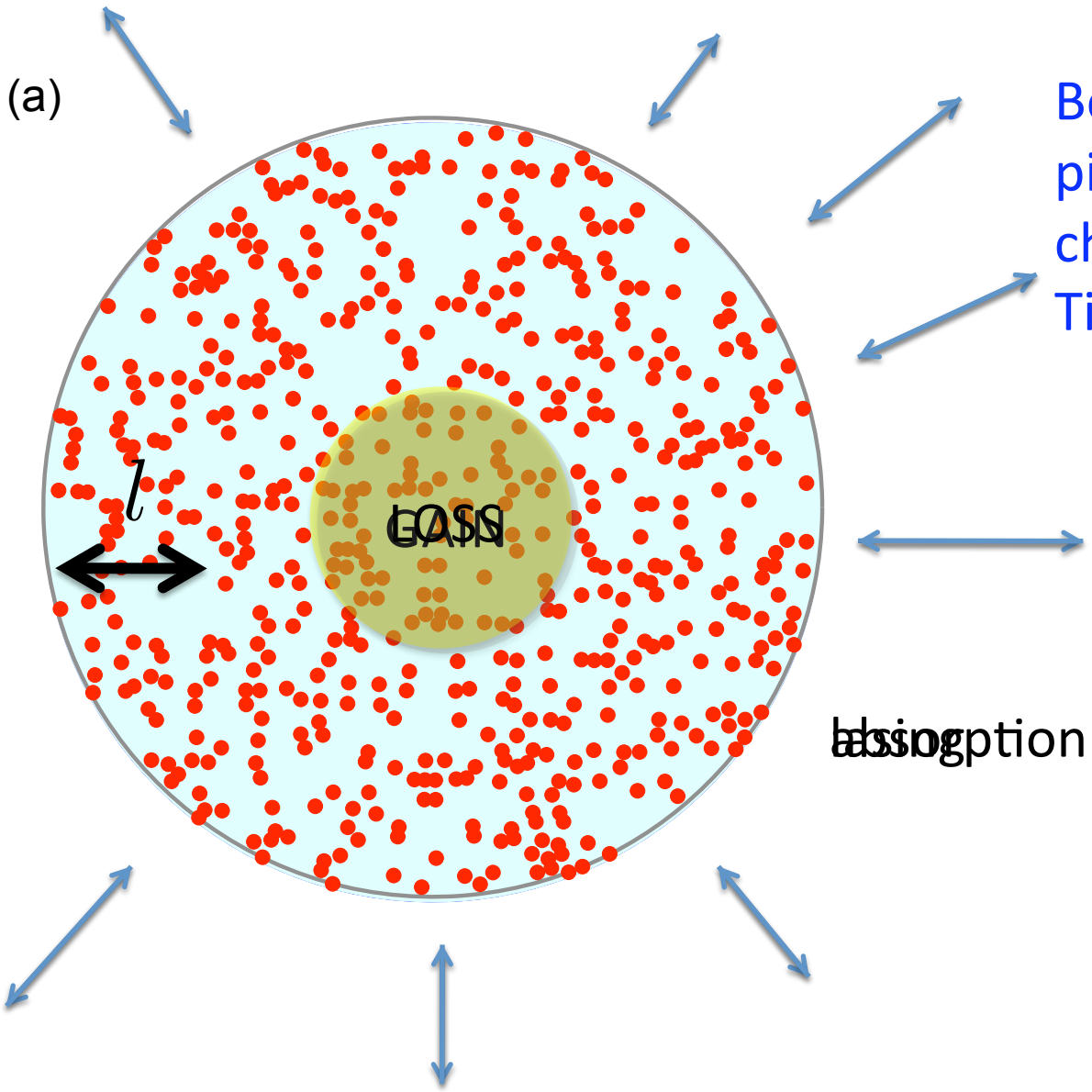
W. Wan, Y. D. Chong, L. Ge, H. Noh, A. D. Stone, H. Cao, *Science*, 2/18/11

"Low-Q Fabry-Perot etalon"

Varying relative phase 0 -> 2π at various λ



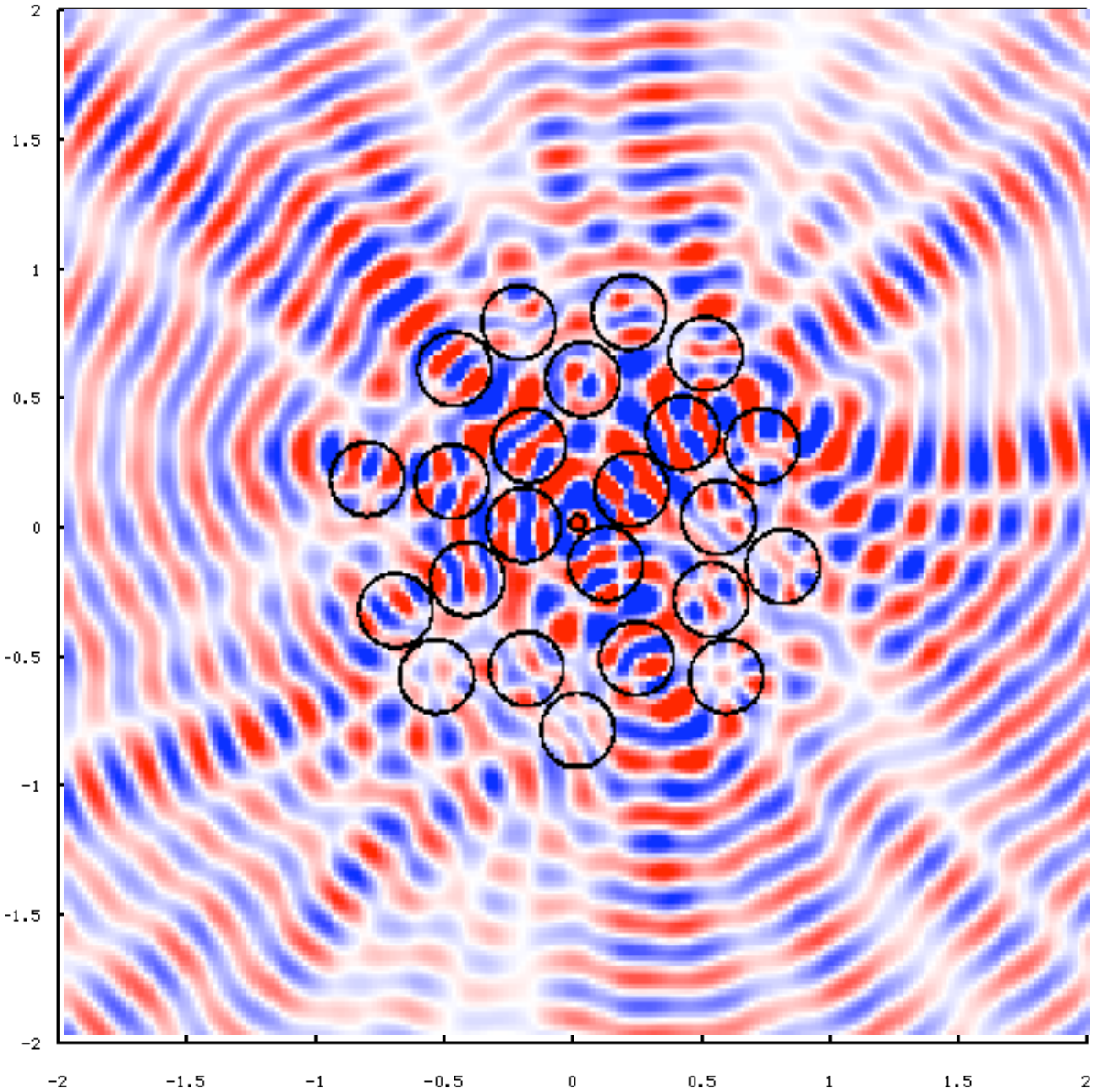
Coherent control of absorption in disordered media



Beyond ray interference picture of CCR, AFP, 2-channel CPA
Time-reverse wave chaos,

Radiation can penetrate a normally opaque outer shell to be fully absorbed in central region!

Sub-wavelength random CPA



Time-reverse of a laser with a highly localized gain medium – “beats” diffraction limit,due to scattering

Sub λ object could note lase – not enough gain

Very lossy objects are easy to find!

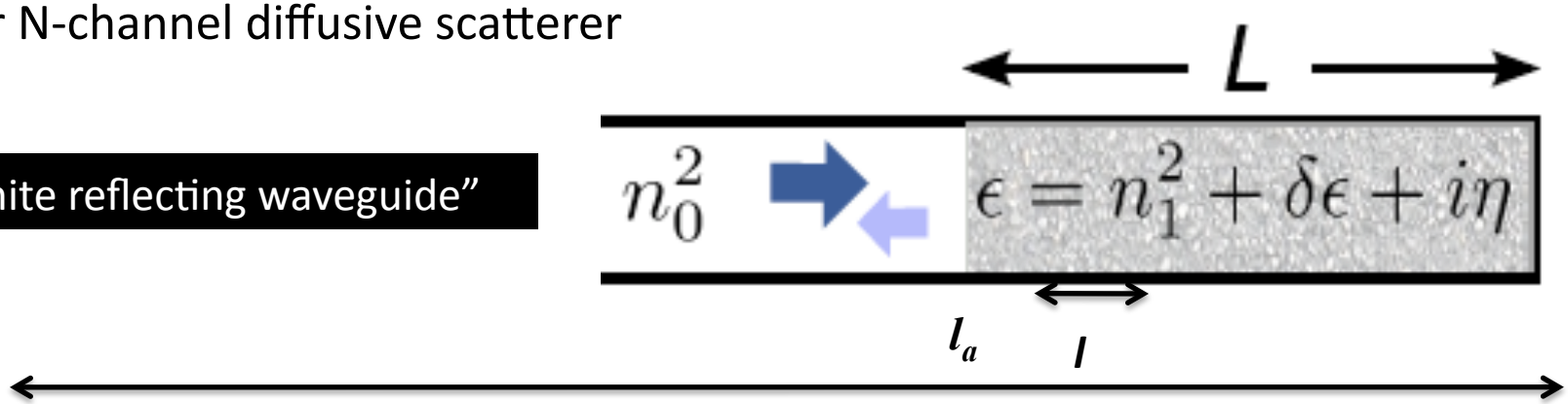
ce n’est pas ”L’effect Fink”. Steady-state, passive sink, narrow band

Beyond CPA: Coherent enhancement of absorption

Y. Chong and A. D. Stone, PRL **107**, 163901 (2011)

- ❑ CPA, like lasing, occurs at discrete points in the (loss, ω) parameter space; if loss can be continuously varied, occurs at discrete narrow ω bands
- ❑ Is it possible to strongly enhance absorption at <u>any</u> ω, if input field tunable?
- ❑ Yes for N-channel diffusive scatterer

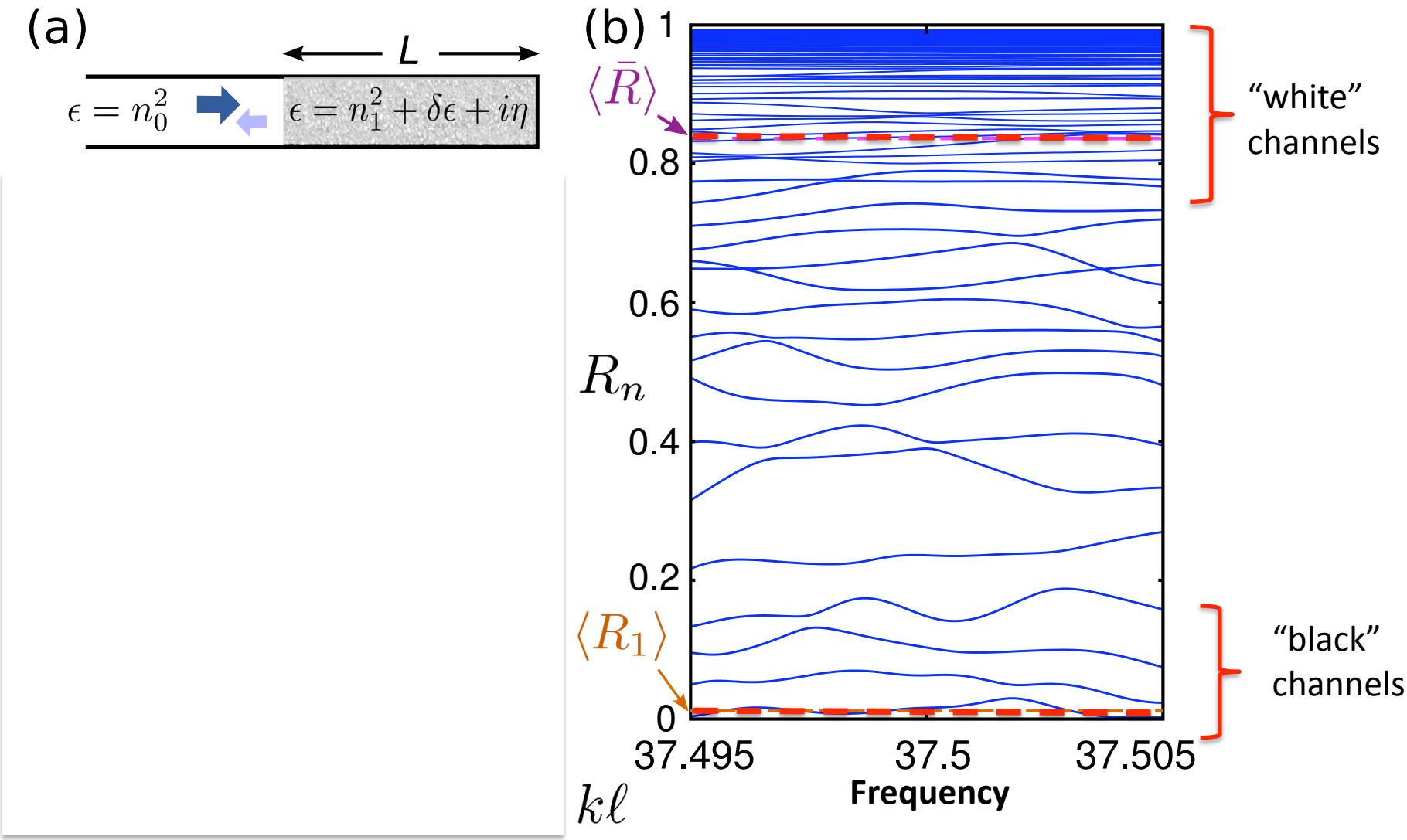
"White reflecting waveguide"



Weak absorptivity can lead to strong absorption with right input wave

- ❑ Study e-value distribution of lossy S-matrix, for reflection geometry can prove J.P.D. is the Laguerre Ensemble of random matrix theory (Beenakker, Chalker, 1996)
- ❑ Can show (Edelman, 1988), that $\langle R_{min} \rangle \sim 1/(2N^2 l/l_a)$; can be much less than one when $\langle R \rangle \sim 1$.

Hidden Black: Coherent Enhancement of Absorption



PT-symmetric wave equations

- Origin: Bender et al. (1998), non-hermitian QM with PT symmetry, {V(x) = V*(-x)}, real spectrum possible
- Exists spontaneous PT-symmetry breaking transitions, spectrum complex, unphysical in QM.
- Physical realization in optics (D. Christodoulides,G. Salamo,D. Kip,M. Segev, focused on waveguides and PT lattices.
- Singularities in PT scattering (simultaneous lasing and coherent perfect absorption) – Longhi, Mostafazadeh, Schomerus

(Longhi, PRA **82** 031801 (2010), Mostafazadeh, PRA **80** 032711 (2009), Schomerus, PRL **104** 233601 (2010))

PT singularities and PT symmetry breaking:
Y.D. Chong, L. Ge, ADS PRL **106**, 093902 (2011).
Generalized Unitarity Relations in 1D scattering:
L. Ge, Y. D. Chong and A. D. Stone, PRA, **85**, 023802 (2012).
$|T-1| = \sqrt{R_l R_R}$ (confirmed in circuits by Kottos group)

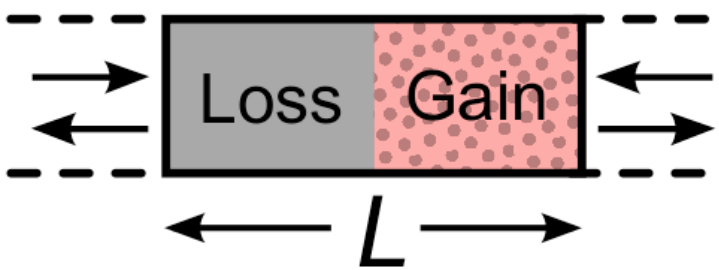
S-matrix approach to PT-structures

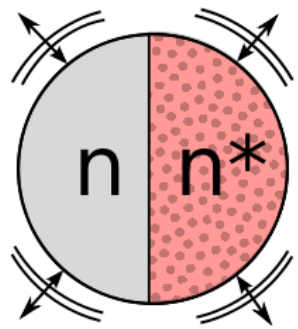
No mapping to paraxial wave equation, assume full wave equation has PT symmetry, analyze resulting S-matrix

$$\mathbf{PT}[n(\vec{r})] = n(\vec{r})$$

1D cases studied:
Cannata et al.,
Mostafazadeh.

PT-heterojunction





PT-disk

$$\mathbf{PT}S(nk)\mathbf{PT} = S^{-1}(nk)$$

Schomerus
Chong et al.

$$\Rightarrow Det(S) = e^{i\theta}$$

But, S is <u>not</u> unitary!

$$\mathbf{S}(nk)\cdot\underline{\psi} = s\underline{\psi} \Rightarrow \mathbf{S}(nk)\cdot\mathbf{PT}\underline{\psi} = \frac{1}{s^*}\mathbf{PT}\underline{\psi}$$

S-matrix eigenstates come in "unimodular pairs": $|s_1 \bullet s_2| = 1$

Spontaneous symmetry-breaking in S-matrix

$$\mathbf{S}(nk) \cdot \underline{\psi} = s\underline{\psi} \Rightarrow \mathbf{S}(nk) \cdot \mathbf{PT}\underline{\psi} = \frac{1}{s^*}\mathbf{PT}\underline{\psi}$$

Two possibilities:

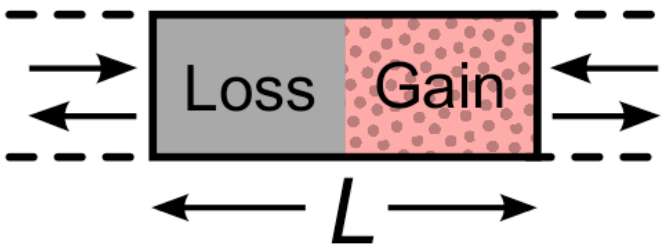
- $\mathbf{PT}\underline{\psi} = c\underline{\psi}$ Scattering eigenstates **preserve** PT symmetry $|s| = 1$

=> PT- symmetric or unimodular "phase"

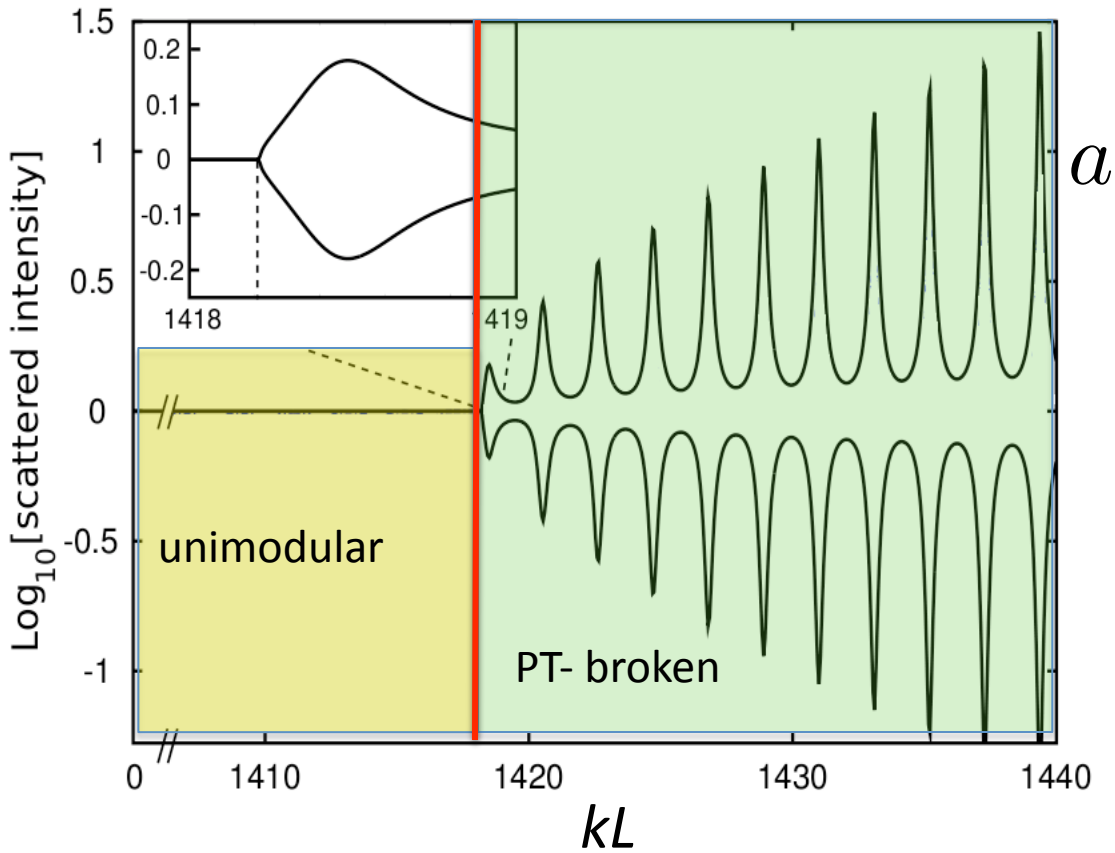
- $\mathbf{PT}\underline{\psi} \neq c\underline{\psi}$ Scattering eigenstates **break** PT symmetry $\frac{1}{s^*} \neq s, |s| \neq 1$

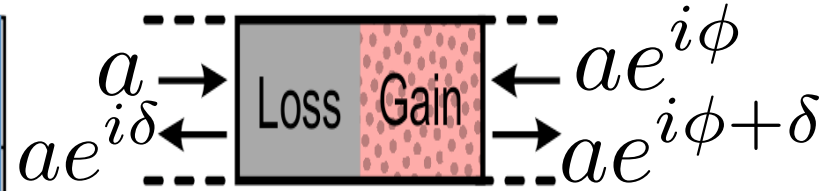
S-matrix eigenstates come in pairs, with $|s_+| = |s_-|^{-1}$, one amplifying, one attenuating – broken symmetry "phase".

Simple 1D PT-symmetric heterojunction



2-channel case, similar to CPA geometry, 2 e-values either uni-modular or amplifying/attenuating





Either amplifying or attenuating => Interferometric amplifier-absorber

Simple HS: CPA on steroids

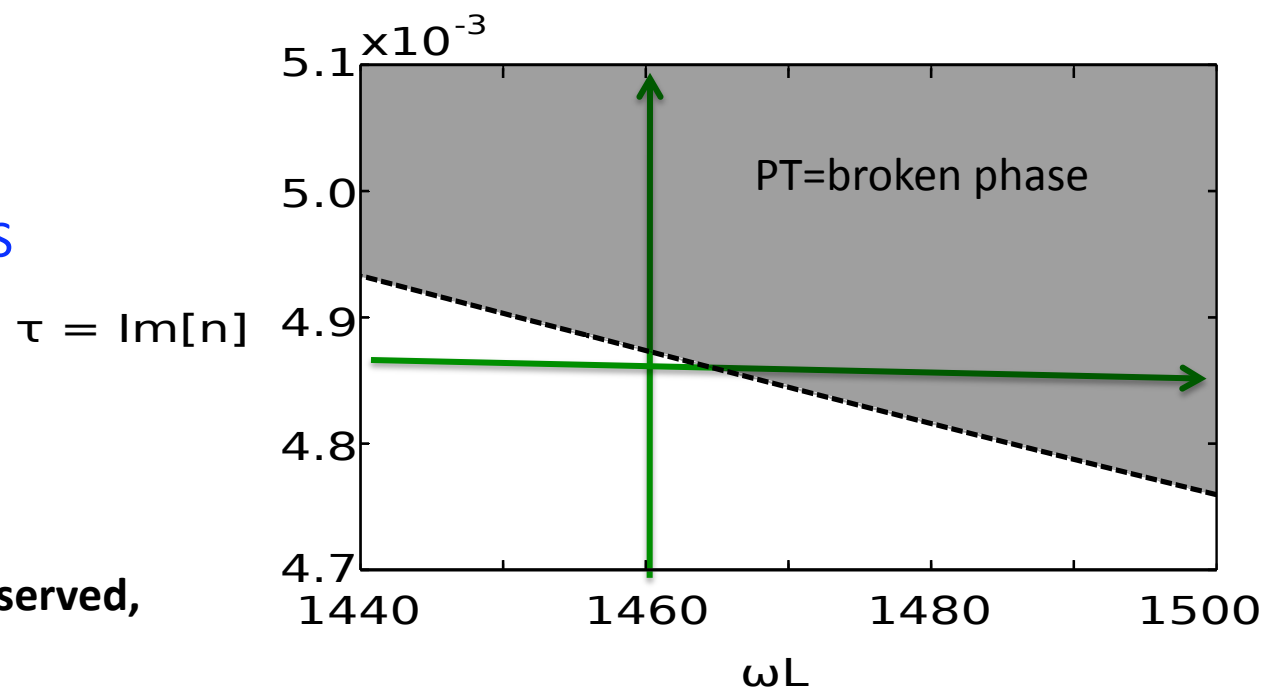
CPA: strongly scattering or perfectly absorbing

PT- CPA: strongly amplifying or absorbing

- ❑ phase transition with increasing n_2 or ω
- ❑ transition at $n_2 \approx 0.005$ at L ≈ 200 µm in simple HS requires a lot of gain!
- ❑ phase boundary depends only weakly

$$\omega_c \approx \ln(2n_0/\tau)c/\tau L.$$

Temporal analog recently observed, Regensburger et al.



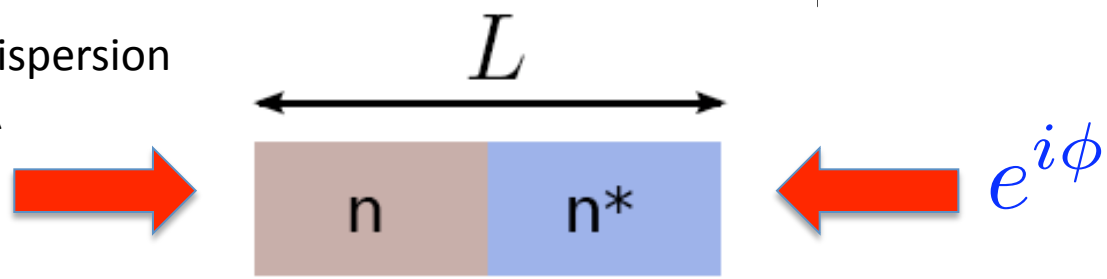
Exact PT "no-go" theorem

Comment: Zyablovsky et al., Arxiv 1205.2820

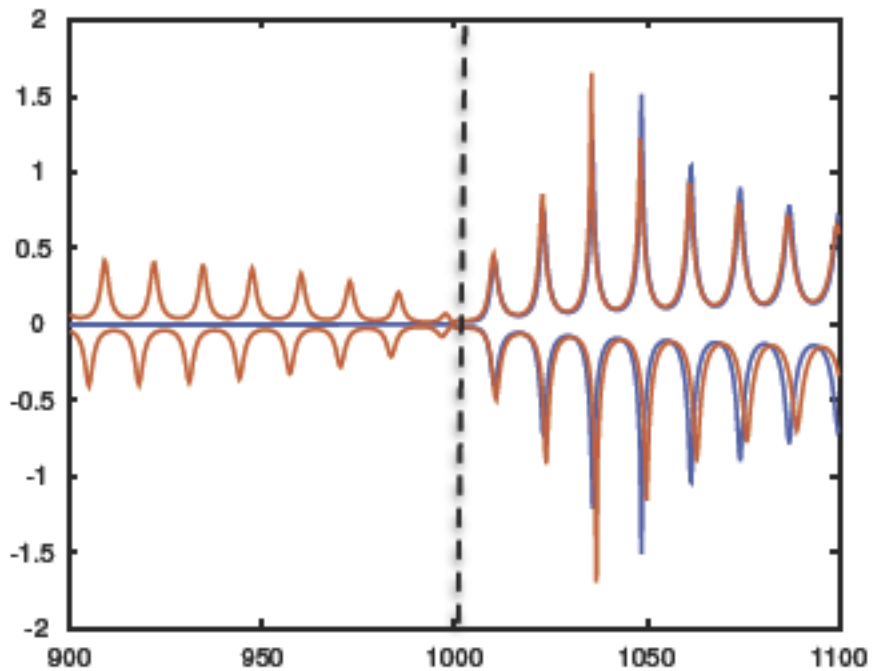
- Proof: $\varepsilon(r,\omega) = \varepsilon^*(-r,\omega)$ not possible over any finite region of the real ω axis (uses analyticity in u.h. plane, uniqueness of analytic continuation)
- PT symmetric S-matrix only exists at finite discrete set of freqs for fixed external parameters (e.g. pump)
- Can only have exact PT transition varying e.g. pump and retuning, not by varying freq at fixed parameters.
- When dispersion is weak can have good approx PT over substantial interval, expt'l problem depends on system.
- At least one expt already show PT behavior as freq as varied (PT microwave electronics, Kottos group).
- Lasing happens at discrete freqs, so exact CPA-laser is possible.

Robustness to dispersion

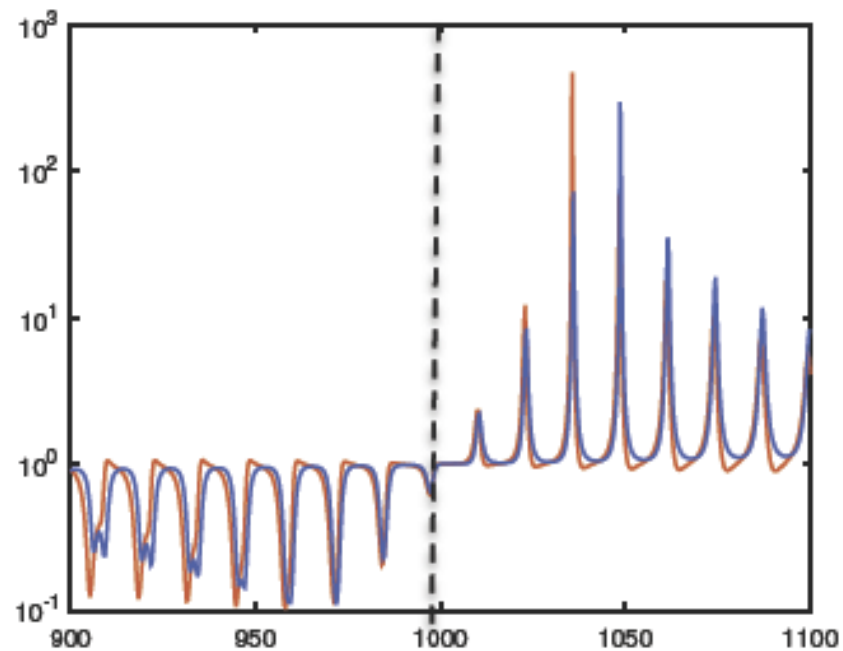
Add 0.1% dispersion
over 10 FSR



eigenvalue "measurement"

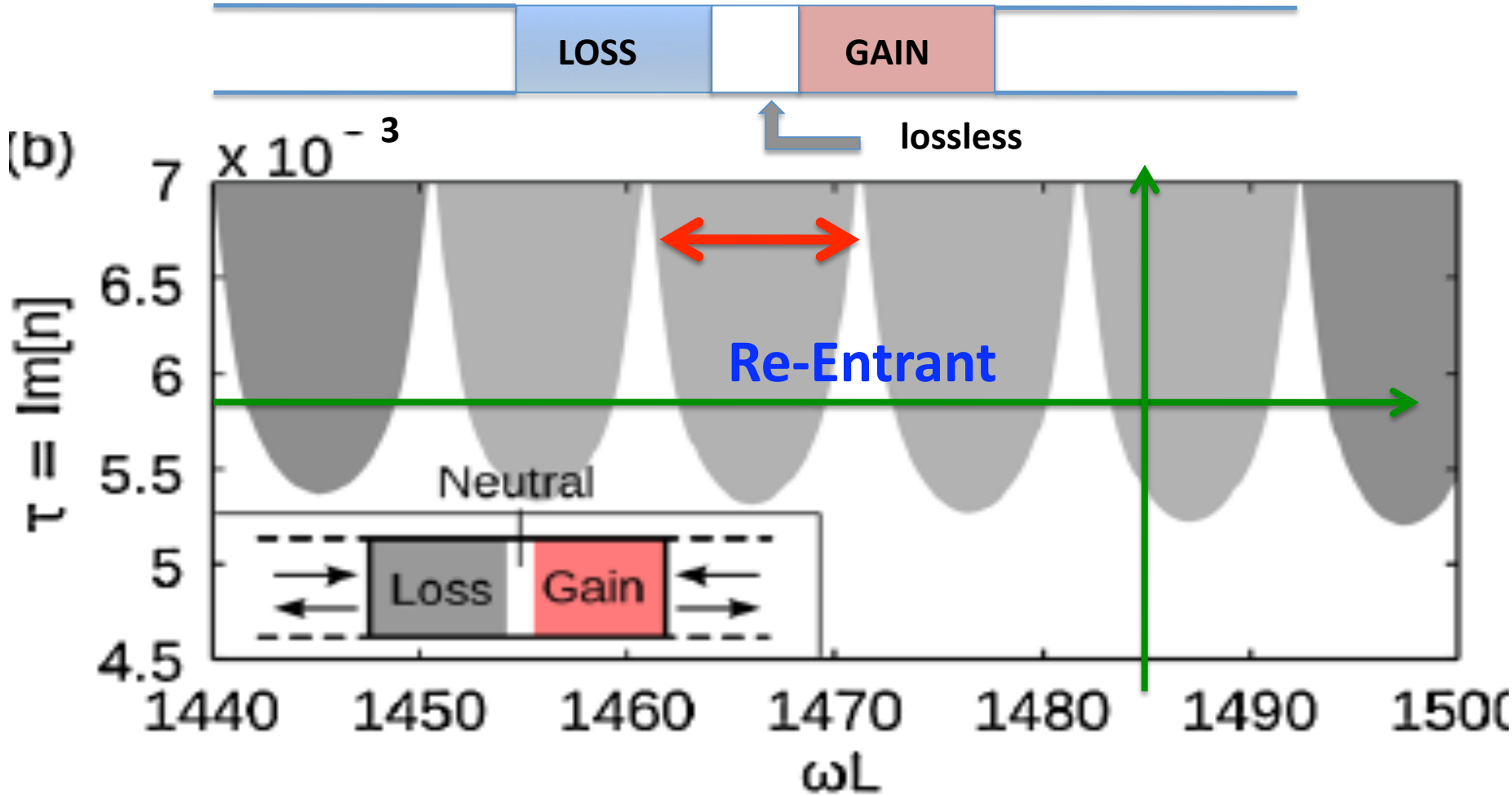


balanced inputs "measurement"



Will assume exact PT symmetry as freq varies henceforth

More complex heterostructures



❑ Re-entrant due to resonance, at transmission resonance perfect coupling suppresses PT transition

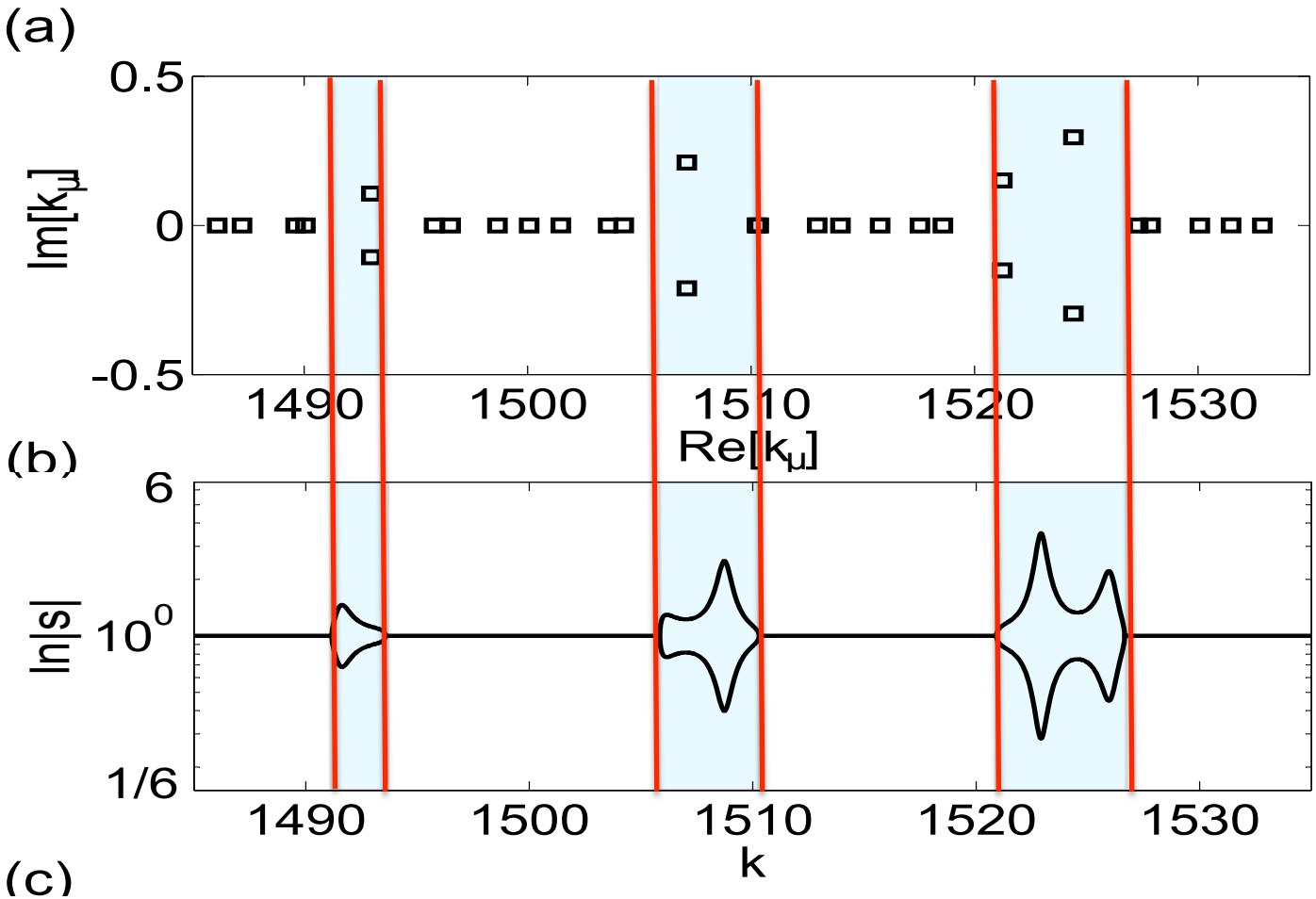
Re-entrant in ω, but not in τ

Enhanced symmetry breaking possible, but trivial

PT transition in S vs. H

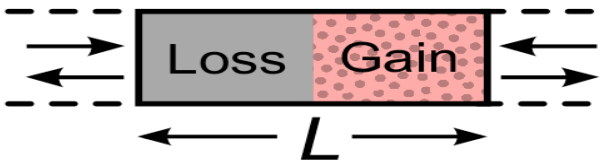
L. Ge, Chong et al, PRA 2012

=> Hamiltonian PT system, discrete real or complex pair e-values

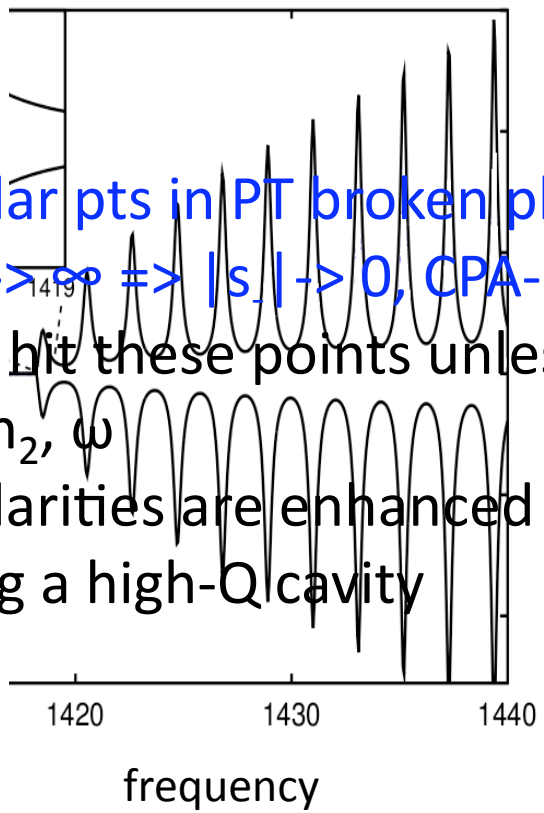


More work on this from Rotter group, e.g. effect of BCs

Singularities in broken PT phase



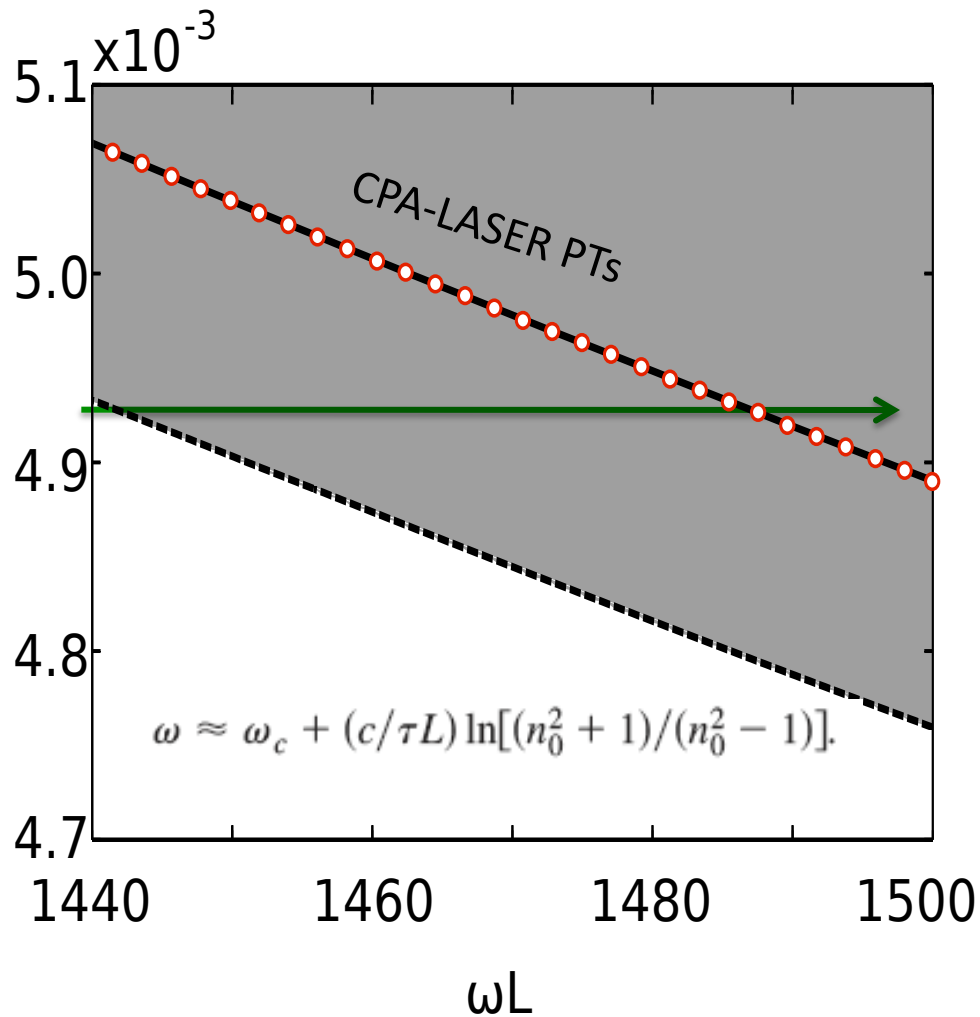
Recall oscillations in broken PT phase of simple heterostructure



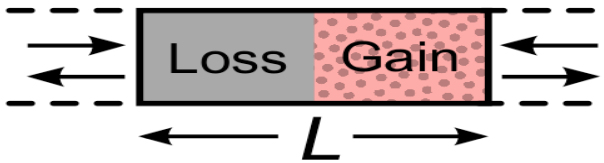
Singular pts in PT broken phase:
$|s_+| \to \infty \Rightarrow |s_-| \to 0$, CPA-Laser

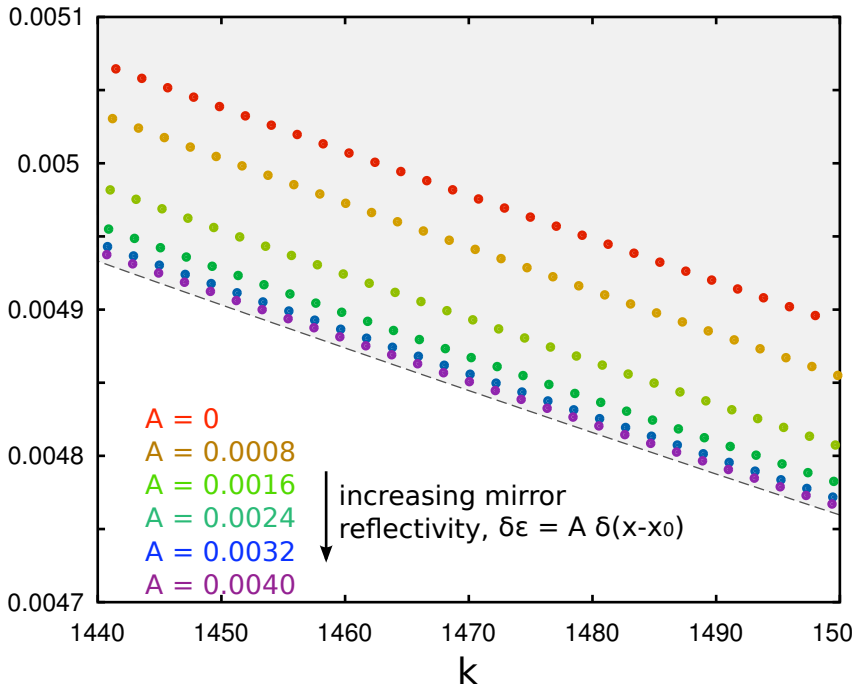
Don't hit these points unless you tune n_2, ω

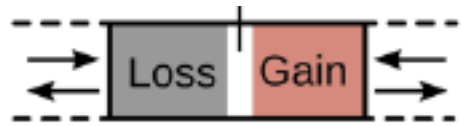
Singularities are enhanced by adding a high-Q cavity

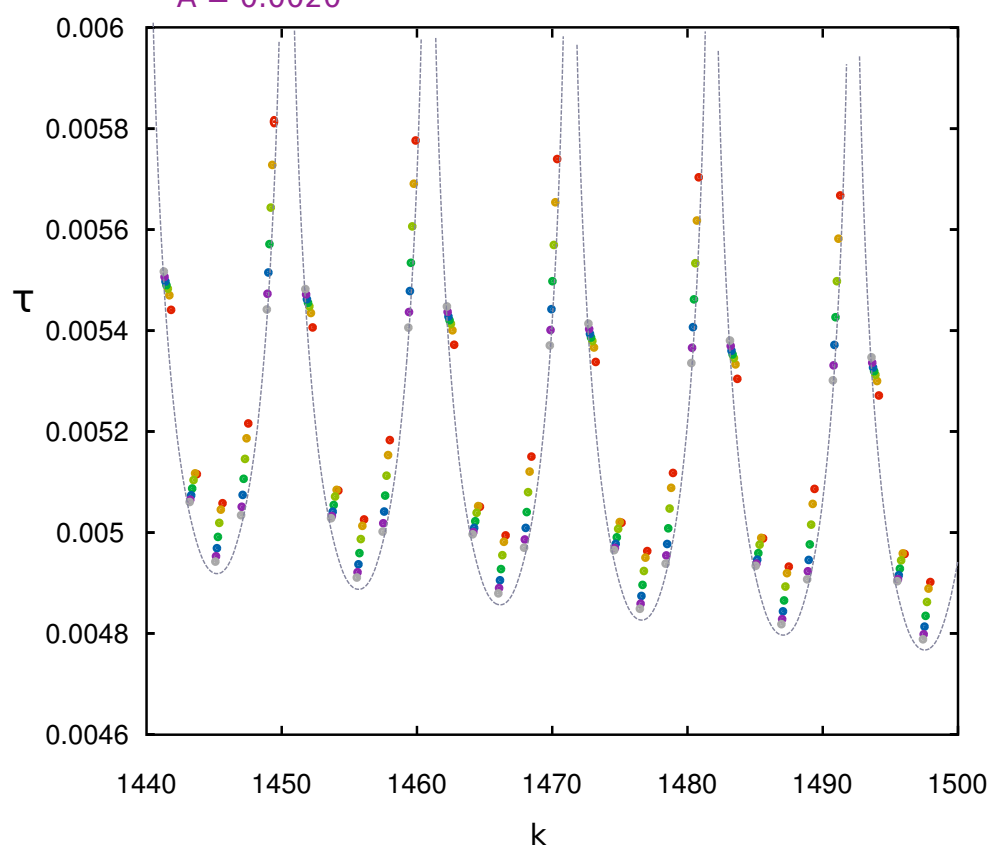


Mirrors move singularities but not the phase boundaries

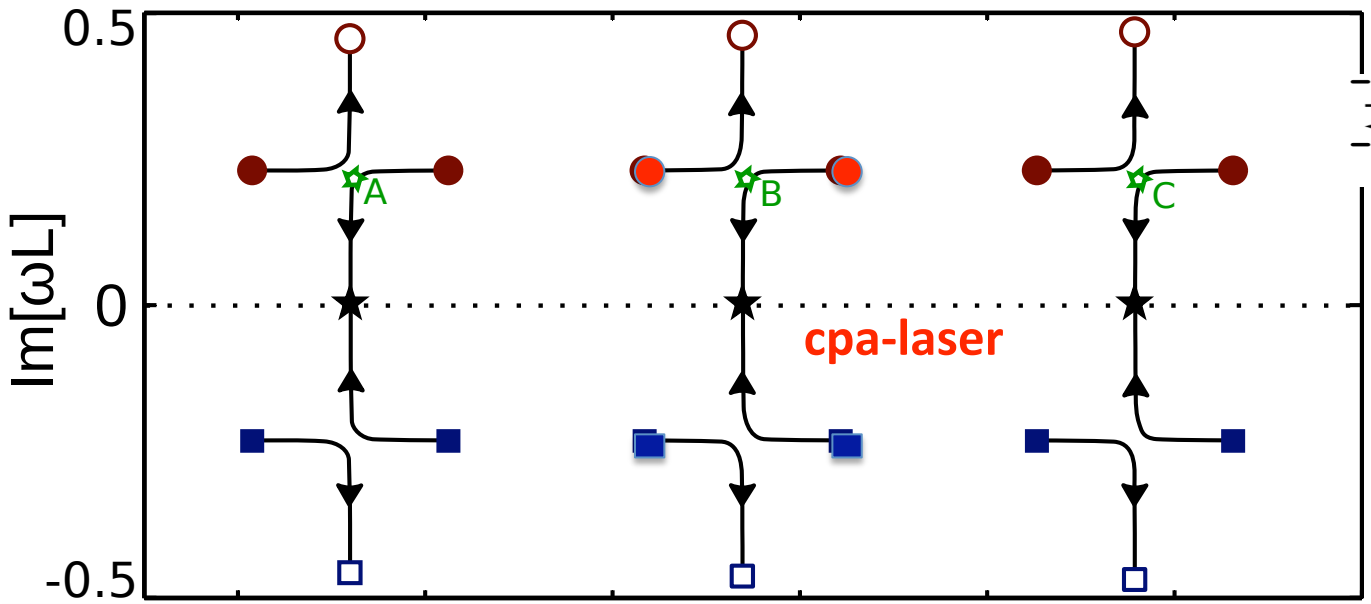


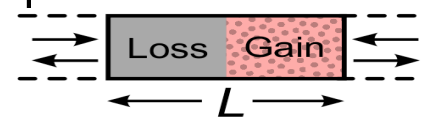






Heterojunction CPA laser singularities: poles and zeros





PT enforces

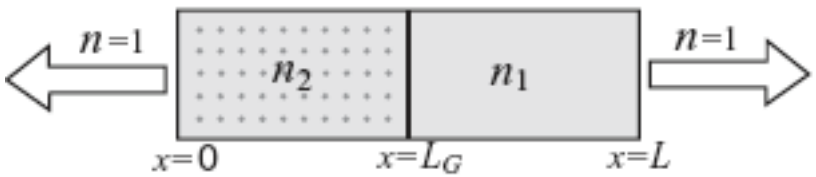
$k_{pole} = k^*_{zero}$

Non-trivial pole motion related to EP

Signature: FSR doubles to 2π/L,resonances shift by ½ an FSR

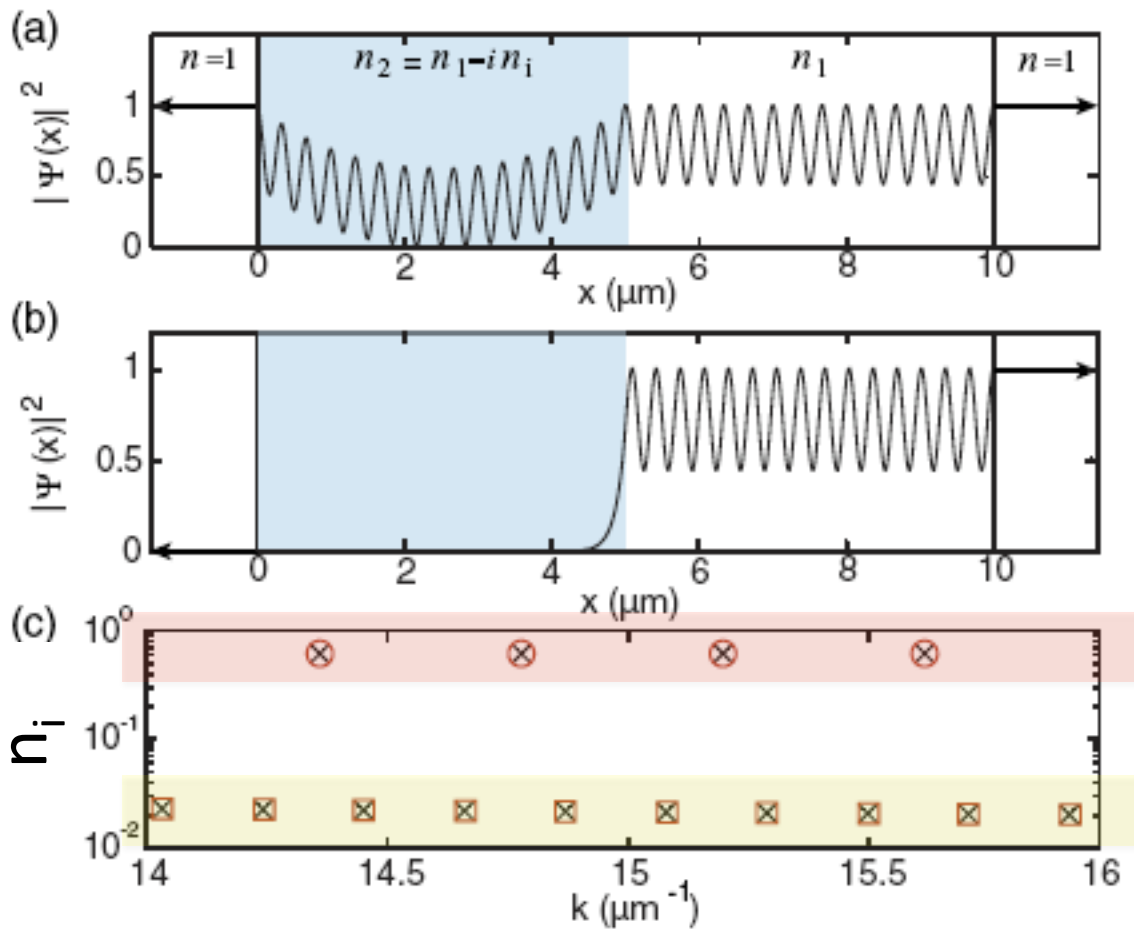
Laser with zero net single-pass gain!

Non-trivial pole motion beyond PT: Partial pumping



Uniform passive dielectric cavity, pumped only in one section

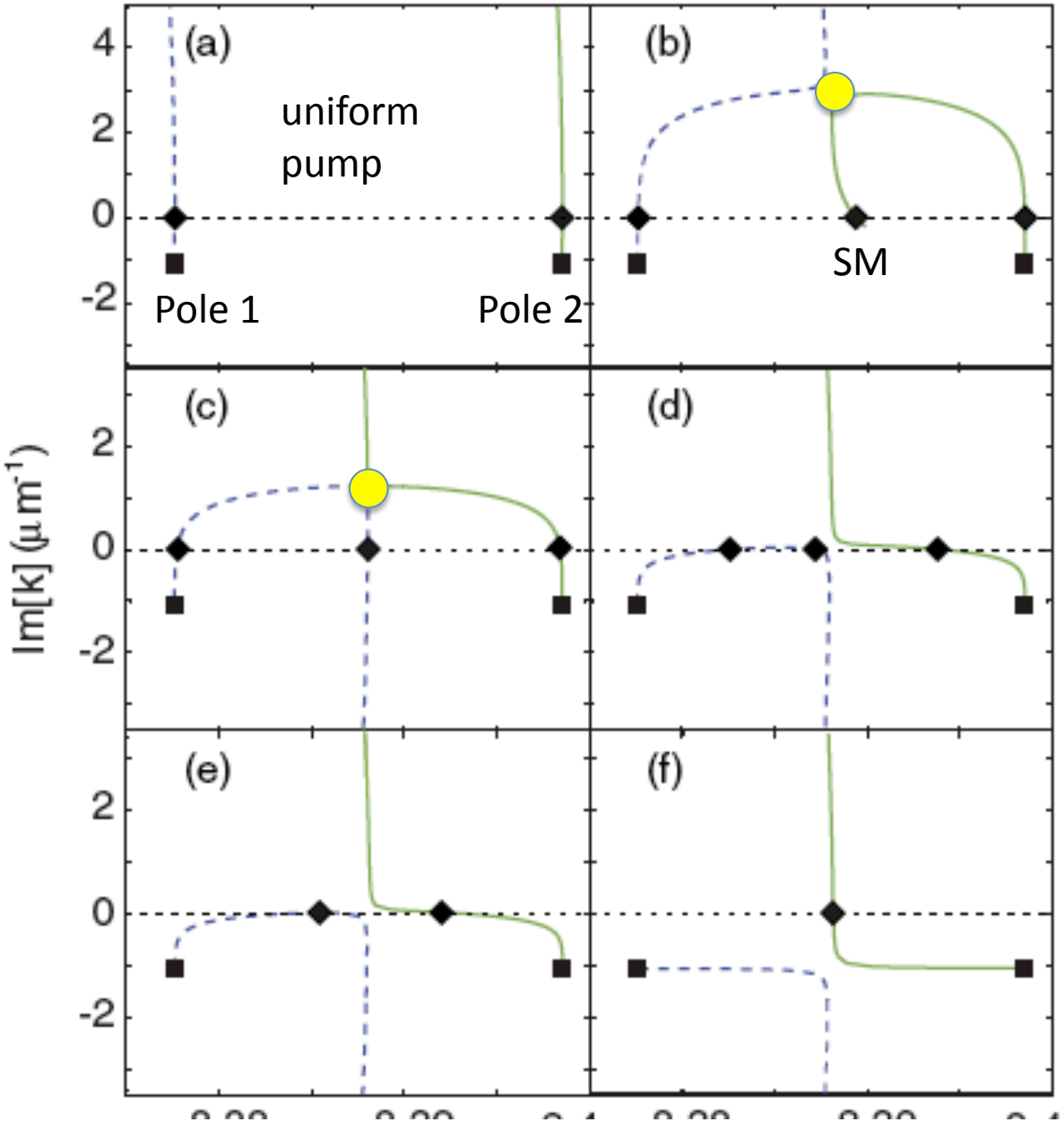
Non-uniform pumping leads to extra lasing modes!
Where do they come from?

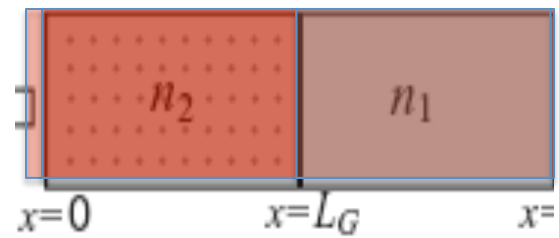


Surface Modes

Conventional Modes

Pole motion and EPs





Exceptional point!

poles coalesce at the EP, role reversal

Two modes associated with pole #1 move closer

Two modes annihilate in inverse bifurcation –> 3 modes reduced to one

gain-loss ratio equal, half modes gone, doubling of FSR, ½ FSR shift (PT)

Summary

- Maxwell operators with gain, loss and gain+ loss give rise to interesting and novel non-hermitian physics.
- Singularities (poles and EPs) control many of the phenomena
- Non-linear ab initio (SALT) theory of lasing, based on S-matrix, quantitative solution of multimode lasing.
- Non-linear S-matrix gives quantum fluctuations as well
- For every laser there is a coherent perfect absorber (related to zeros of S-matrix) – can generalize to coherently enhance absorption
- PT S-matrix, poles and zeros coincide on real axis: CPA-Laser
- Non-trivial pole motion and EPs give rise to novel modes